On the universality of Estermann zeta-functions

R. Garunkštis, A. Laurinćikas, R. Šleževičienė*, J. Steuding†

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Dedicated to Prof. Wolfgang Schwarz at the occasion of his retirement.

Abstract. We prove a universality property for certain Estermann zeta-functions and state some consequences concerning the value distribution and functional independence.

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1 Introduction

For every complex number α we define the generalized divisor function by

$$\sigma_\alpha(n) = \sum_{d|n} d^\alpha.$$

It is well-known that $\sigma_0(n) \ll n^\epsilon$ for all $n \in \mathbb{N}$ and any $\epsilon > 0$ (see, for example, [1]); here and in the sequel we write $f(x) = O(g(x))$ and $f(x) \ll g(x)$, resp., when $\limsup_{x \rightarrow \infty} \frac{|f(x)|}{g(x)}$ exists. Since

$$\sigma_\alpha(n) = n^\alpha \sigma_{-\alpha}(n), \tag{1}$$

*Department of Mathematics and Informatics, Vilnius University, Naugarduko 24, 2600 Vilnius, Lithuania, ramunas.garunkstis@maf.vu.lt, antanas.laurincikas@maf.vu.lt, rasa.slezeviciene@maf.vu.lt

†Mathematisches Seminar, Johann Wolfgang Goethe-Universität Frankfurt, Robert-Mayer-Str. 10, 60054 Frankfurt, Germany, steuding@math.uni-frankfurt.de

it follows that $\sigma_\alpha(n) \ll n^{\epsilon + \max\{\operatorname{Re} \alpha, 0\}}$. In the sequel let k and ℓ be coprime integers, and denote by $s = \sigma + it$, as usual, a complex variable. Then, for $\sigma > \max\{1 + \operatorname{Re} \alpha, 1\}$, the Estermann zeta-function is given by

$$E\left(s; \frac{k}{\ell}, \alpha\right) = \sum_{n=1}^{\infty} \frac{\sigma_\alpha(n)}{n^s} \exp\left(2\pi i n \frac{k}{\ell}\right). \quad (2)$$

$E(s; \frac{k}{\ell}, \alpha)$ can be analytically continued to a meromorphic function, which is regular in the whole complex plane up to two simple poles at $s = 1$ and $s = 1 + \alpha$ if $\alpha \neq 0$, and one double pole at $s = 1$ if $\alpha = 0$, respectively. Further, $E(s; \frac{k}{\ell}, \alpha)$ satisfies the functional equation

$$\begin{aligned} E\left(s; \frac{k}{\ell}, \alpha\right) &= \frac{1}{\pi} \left(\frac{2\pi}{\ell}\right)^{2s-1-\alpha} \Gamma(1-s) \Gamma(1+\alpha-s) \left(\cos\left(\frac{\pi\alpha}{2}\right) E\left(1+\alpha-s; \frac{\bar{k}}{\ell}, \alpha\right) \right. \\ &\quad \left. - \cos\left(\pi s - \frac{\pi\alpha}{2}\right) E\left(1+\alpha-s; -\frac{\bar{k}}{\ell}, \alpha\right) \right), \end{aligned} \quad (3)$$

where $\bar{k}$ is defined by $k\bar{k} \equiv 1 \pmod{\ell}$. For $\alpha = 0$ the Estermann zeta-function was introduced by Estermann [3], and for $\alpha \in (-1, 0]$ by Kiuchi [5]. Since $\sigma_\alpha(n)$ is an entire function in α , the statements above hold by analytic continuation for any complex α .

The aim of this paper is to prove a universality property for Estermann zeta-functions, similar to the celebrated one for the Riemann zeta-function. For this purpose we exchange in (2) the additive character $\exp(2\pi i n \frac{k}{\ell})$ by a multiplicative character $\chi \pmod{\ell}$, and define, for $\sigma > \max\{1 + \operatorname{Re} \alpha, 1\}$,

$$E(s; \chi, \alpha) = \sum_{n=1}^{\infty} \frac{\sigma_\alpha(n)}{n^s} \chi(n). \quad (4)$$

2 Preliminaries

Let φ be Euler's totient. Further, denote the Gauss sum associated to a Dirichlet character $\chi \pmod{\ell}$ by

$$\tau(\chi) = \sum_{j \pmod{\ell}} \chi(j) \exp\left(2\pi i \frac{j}{\ell}\right).$$

The following lemma links additive and multiplicative characters.

Lemma 1 *Let m, ℓ be coprime integers, and let χ be a character $\pmod{\ell}$. Then*

$$\begin{aligned} \exp\left(2\pi i \frac{m}{\ell}\right) &= \frac{1}{\varphi(\ell)} \sum_{\chi \pmod{\ell}} \tau(\bar{\chi}) \chi(m), \\ \chi(m) \tau(\bar{\chi}) &= \sum_{j \pmod{\ell}} \bar{\chi}(j) \exp\left(2\pi i \frac{j m}{\ell}\right); \end{aligned}$$

the latter formula holds without any restriction on m and ℓ if χ is primitive.

Proof. For the proof of the second formula see [1]; the first formula follows easily from the second one: since

$$\begin{aligned} \sum_{\chi \bmod \ell} \tau(\bar{\chi})\chi(m) &= \sum_{\chi \bmod \ell} \sum_{j \bmod \ell} \bar{\chi}(j) \exp\left(2\pi i \frac{jm}{\ell}\right) \\ &= \sum_{j \bmod \ell} \exp\left(2\pi i \frac{jm}{\ell}\right) \sum_{\chi \bmod \ell} \bar{\chi}(j), \end{aligned}$$

the well-known orthogonality relation for characters implies the formula. •

This leads to a representation of Estermann zeta-functions by the functions $E(s, \chi, \alpha)$ given by (4) and vice versa. Recall that the Dirichlet L -functions attached to characters $\chi \bmod \ell$ is, for $\sigma > 1$, defined by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1},$$

here and in the sequel p stands always for a prime number and χ_0 denotes the principal character $\bmod \ell$. We may interpret the Riemann zeta-function $\zeta(s)$ as the Dirichlet L -function to the principal character $\chi_0 \bmod 1$.

Theorem 2 *Let ℓ be prime. Then*

$$E\left(s; \frac{k}{\ell}, \alpha\right) = \frac{1}{\varphi(\ell)} \sum_{\substack{\chi \bmod \ell \\ \chi \neq \chi_0}} \tau(\bar{\chi})\chi(k)E(s; \chi, \alpha) + \Lambda(s; \alpha)E(s; \chi_0, \alpha),$$

where

$$\Lambda(s; \alpha) := \begin{cases} \frac{2\ell - \ell^{1-s} - \ell^s}{\ell^s(\ell-1)(1-\ell^{-s})^2} & \text{if } \alpha = 0, \\ \frac{\ell - \ell^{1+\alpha-s} - \ell^{1+2\alpha} + \ell^{1+2\alpha-s} - \ell^s + \ell^{\alpha+s}}{\ell^s(\ell-1)(1-\ell^\alpha)(1-\ell^{-s})(1-\ell^{\alpha-s})} & \text{otherwise.} \end{cases} \quad (5)$$

Proof. Since ℓ is prime, all non-principal characters $\chi \bmod \ell$ are primitive. Hence, Lemma 1 gives, for $\sigma > \max\{1 + \operatorname{Re} \alpha, 1\}$,

$$\begin{aligned} E\left(s; \frac{k}{\ell}, \alpha\right) &= \sum_{\substack{n=1 \\ n \not\equiv 0 \bmod \ell}}^{\infty} \frac{\sigma_\alpha(n)}{n^s} \frac{1}{\varphi(\ell)} \sum_{\chi \bmod \ell} \tau(\bar{\chi})\chi(nk) + \sum_{\substack{n=1 \\ n \equiv 0 \bmod \ell}}^{\infty} \frac{\sigma_\alpha(n)}{n^s} \\ &= \frac{1}{\varphi(\ell)} \sum_{\chi \bmod \ell} \tau(\bar{\chi})\chi(k) \sum_{n=1}^{\infty} \frac{\sigma_\alpha(n)\chi(n)}{n^s} + \frac{1}{\ell^s} \sum_{m=1}^{\infty} \frac{\sigma_\alpha(m\ell)}{m^s}. \end{aligned} \quad (6)$$

It is well-known (formulae 1.4.1,2 of [13]) that

$$\sum_{m=1}^{\infty} \frac{\sigma_\alpha(m\ell)}{m^s} = \begin{cases} \zeta(s)^2 (2 - \ell^{-s}) & \text{if } \alpha = 0, \\ \zeta(s)\zeta(s - \alpha) \frac{1 - \ell^{\alpha-s} - \ell^{2\alpha} + \ell^{2\alpha-s}}{1 - \ell^\alpha} & \text{otherwise.} \end{cases}$$

This yields

$$E(s; \chi_0, \alpha) = \sum_{\substack{n=1 \\ n \not\equiv 0 \pmod{\ell}}}^{\infty} \frac{\sigma_{\alpha}(n)}{n^s} = \begin{cases} \zeta(s)^2 \left(1 - \frac{2-\ell^{-s}}{\ell^s}\right) & \text{if } \alpha = 0, \\ \zeta(s)\zeta(s-\alpha) \left(1 - \frac{1-\ell^{\alpha-s}-\ell^{2\alpha}+\ell^{2\alpha-s}}{\ell^s(1-\ell^{\alpha})}\right) & \text{otherwise.} \end{cases} \quad (7)$$

Substituting this in formula (6), the assertion follows (after a short computation) for $\sigma > \max\{1 + \operatorname{Re} \alpha, 1\}$, and by analytic continuation elsewhere. •

An arithmetic function $f : \mathbb{N} \rightarrow \mathbb{C}$ is called multiplicative if $f(m \cdot n) = f(m) \cdot f(n)$ for all coprime m, n . As a Dirichlet series with multiplicative coefficients $E(s; \chi, \alpha)$ possesses an Euler product, i.e. a representation as a product taken over primes.

Theorem 3 *Let ℓ be prime and χ be a character mod ℓ . Then*

$$E(s; \chi, \alpha) = \frac{1}{\tau(\bar{\chi})} \sum_{j \pmod{\ell}} \bar{\chi}(j) E\left(s; \frac{j}{\ell}, \alpha\right)$$

if $\chi \neq \chi_0$, and otherwise

$$E(s; \chi_0, \alpha) = \begin{cases} (1 - \ell^{-s})^2 E(s; 1, 0) & \text{if } \alpha = 0, \\ \frac{\ell^s - \ell^{s+\alpha} - 1 + \ell^{\alpha-s} + \ell^{2\alpha} - \ell^{2\alpha-s}}{\ell^s(1-\ell^{\alpha})} E(s; 1, \alpha) & \text{otherwise.} \end{cases}$$

In any case,

$$\begin{aligned} E(s; \chi, \alpha) &= \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \left(1 - \frac{\chi(p)}{p^{s-\alpha}}\right)^{-1} \\ &= L(s, \chi) L(s - \alpha, \chi); \end{aligned}$$

the Euler product representation is valid for $\sigma > \max\{1 + \operatorname{Re} \alpha, 1\}$ while the latter formula holds throughout the whole complex plane. In particular, $E(s; \chi, \alpha)$ is an entire function if χ is non-principal, whereas $E(s; \chi_0, \alpha)$ is analytic in $\mathbb{C}$ except for poles at $s = 1, 1 + \alpha$.

Proof. The representations of $E(s; \chi, \alpha)$ via Estermann zeta-functions are immediate consequences of Lemma 1 and (7).

For the second assertion note that

$$\sigma_{\alpha}(p^j) = 1 + p^{\alpha} + p^{\alpha j} = \begin{cases} 1 + j & \text{if } \alpha = 0, \\ \frac{1 - p^{\alpha(j+1)}}{1 - p^{\alpha}} & \text{otherwise.} \end{cases}$$

Moreover, $\sigma_{\alpha}(n)$ is multiplicative, and together with $\chi(p^k) = \chi(p)^k$ the Euler product representation follows after some easy computations, valid in the half plane of absolute convergence of the series representation (4). This yields the representation of $E(s; \chi, \alpha)$ as a product of Dirichlet L -functions, by analytic continuation valid throughout $\mathbb{C}$. The analyticity is a simple consequence of well-known properties of Dirichlet L -fuctions. •

For the sake of completeness note that, as a simple consequence of the functional equation for Dirichlet L -functions, $E(s; \chi, \alpha)$ satisfies

$$E(s; \chi, \alpha) = \frac{\tau(\chi)^2}{\pi \ell} \left(\frac{2\pi}{\ell} \right)^{2s-1-\alpha} \Gamma(1-s) \Gamma(1+\alpha-s) E(1+\alpha-s; \bar{\chi}, \alpha) \times \\ \times \left(\cos \left(\frac{\pi(2-2s+\alpha)}{2} \right) + \chi(-1) \cos \left(\frac{\pi\alpha}{2} \right) \right),$$

which should be compared with the functional equation for Estermann zeta-functions (3). In a sense, the functions $E(s; \chi, \alpha)$ have *better* analytic properties than Estermann zeta-functions. We know this phenomenon from the Riemann zeta-function with its multiplicative twists (Dirichlet L -functions) and its additive twists (Lerch zeta-functions).

By (1) we have further functional equations

$$E\left(s; \frac{k}{\ell}, \alpha\right) = E\left(s - \alpha; \frac{k}{\ell}, -\alpha\right) \quad \text{and} \quad E(s; \chi, \alpha) = E(s - \alpha; \chi, -\alpha).$$

Therefore, it is possible to reflect the following results via $\alpha \mapsto -\alpha$. In the sequel we consider only α with negative real part $a := \operatorname{Re} \alpha$.

3 Universality

In 1975 Voronin [14] proved a remarkable universality property of the Riemann zeta-function, namely that any non-vanishing analytic function $g(s)$, defined on the disc $\{s \in \mathbb{C} : |s| < \frac{1}{4}\}$, and continuous on the boundary, can be approximated to any degree of accuracy by $\zeta(s)$ somewhere in the strip $\frac{1}{2} < \sigma < 1$. Later, it was even proved that the set of translations of $\zeta(s)$ which approximate a given function $g(s)$ has a positive lower (Lebesgue-) density; see [6]. This result was extended, for example, to joint universality for Dirichlet L -functions by Voronin himself [14], to universality for Dedekind zeta-functions by Reich [9], for Hurwitz zeta-functions by Bagchi [2], for Dirichlet series with multiplicative coefficients by Laurinćikas [6], and for more general Dirichlet series by Gonek [4], Laurinćikas, Schwarz and Steuding [7]. The Linnik-Ibragimov conjecture states that all functions given by Dirichlet series, analytically continuable to the left of the half plane of absolute convergence, and which satisfy some natural growth conditions, are universal.

We add to the known examples of Dirichlet series with the universality property certain Estermann zeta-functions.

Theorem 4 *Suppose that $a < -1$, $\ell \geq 5$ is prime, and that*

$$\sum_{j=1}^{\infty} \frac{|\sigma_{\alpha}(p^j)|}{p^{j\beta}} < 1 \tag{8}$$

holds for $p = 2, 3$ with some $\beta \in (\frac{1}{2}, 1)$. Let K be a compact subset of the vertical strip $\beta < \sigma < 1$ with connected complement, and let $g(s)$ be a continuous function on K which is holomorphic in the interior of K . Then, for any $\epsilon > 0$,

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \max_{s \in K} \left| E\left(s + i\tau; \frac{k}{\ell}, \alpha\right) - g(s) \right| < \epsilon \right\} > 0.$$

The proof of this theorem is based on a joint universality result for $E(s; \chi, \alpha)$. Therefore, we have to quote a result from [11].

We say that a multiplicative function f belongs to the class $\mathcal{M}_\chi = \mathcal{M}_\chi(c_1, \beta, \theta, c_2)$ if the following axioms are satisfied:

- (i) there exists a constant c_1 such that $|f(n)| \leq c_1$ for all $n \in \mathbb{N}$;
- (ii) there exists a constant $\beta \in (\frac{1}{2}, 1)$ such that the Dirichlet series

$$Z(s, f) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

has an analytic continuation of finite order to the half plane $\sigma \geq \beta$, except for at most one simple pole at $s = 1$;

- (iii) for $\sigma \geq \beta$ the estimate

$$\int_0^T |Z(\sigma + it, f)|^2 dt \ll T$$

holds, as $T \rightarrow \infty$;

- (iv) for all $1 \leq h \leq m$, which are coprime to m , exists a positive constant θ_h such that

$$\lim_{x \rightarrow \infty} \frac{1}{\pi(x; h, m)} \sum_{\substack{p \leq x \\ p \equiv h \pmod{m}}} |f(p)|^2 = \theta_h,$$

where the sum is taken over all prime numbers $p \leq x$, and where $\pi(x; h, m)$ counts the primes $p \leq x$ in the residue class $h \pmod{m}$;

- (v) there exists a constant $c_2 < 1$ such that the estimate

$$\sum_{j=1}^{\infty} \frac{|f(p^j)|}{p^{j\beta}} \leq c_2$$

holds for all primes p .

We consider Dirichlet series with multiplicative coefficients $f(n)$ twisted by a Dirichlet character χ , given by

$$Z(s, f\chi) = \sum_{n=1}^{\infty} \frac{f(n)\chi(n)}{n^s} = \prod_p \left(1 + \sum_{j=1}^{\infty} \frac{f(p^j)\chi(p^j)}{p^{js}} \right).$$

In [11] the following theorem was proved.

Theorem 5 *Suppose that $\chi_1, \dots, \chi_N$ are pairwise non-equivalent characters, and that $f\chi_j \in \mathcal{M}_\chi$ for $1 \leq j \leq N$. For $1 \leq j \leq N$ let K_j be a compact subset of $D := \{s \in \mathbb{C} : \beta < \sigma < 1\}$ with connected complement, and $g_j(s)$ be a continuous non-vanishing function on K_j which is analytic in the interior of K_j . Then, for any $\epsilon > 0$,*

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \max_{1 \leq j \leq n} \max_{s \in K_j} |Z(s + i\tau, f\chi_j) - g_j(s)| < \epsilon \right\} > 0.$$

From this joint universality we will deduce

Corollary 6 *Suppose that $\text{Re } \alpha = a < -1$, $\ell \geq 5$ is prime, and that condition (8) is fulfilled for $p = 2, 3$ with some $\beta \in (\frac{1}{2}, 1)$. For $1 \leq j \leq n$ let χ_j be pairwise non-equivalent characters mod ℓ , let K_j be a compact subset of D with connected complement, and let $g_j(s)$ be a continuous non-vanishing function on K_j which is analytic in the interior of K_j . Then, for any $\epsilon > 0$,*

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \max_{1 \leq j \leq n} \max_{s \in K_j} |E(s + i\tau, \chi_j, \alpha) - g_j(s)| < \epsilon \right\} > 0.$$

Proof. Obviously, $\sigma_\alpha(n)$ is multiplicative. Since for $a < -1$,

$$|\sigma_\alpha(n)\chi(n)| \leq \sigma_a(n) \leq \sum_{d \leq n} d^a < \zeta(-a),$$

condition (i) holds with $c_1 = \zeta(-a)$. In view to Theorem 3 condition (ii) is fulfilled for any $\beta \in (\frac{1}{2}, 1)$. Note that $L(s - \alpha, \chi)$ is bounded in the half plane $\sigma > 1 + a$. Thus, by the representation of Theorem 3, standard mean square estimates for Dirichlet L -functions (see, for example, [8]) imply

$$\begin{aligned} \int_0^T |E(\sigma + it; \chi, \alpha)|^2 dt &\ll \int_0^T |L(\sigma + it, \chi)|^2 |L(\sigma - \alpha + it, \chi)|^2 dt \\ &\ll \int_0^T |L(\sigma + it, \chi)|^2 dt \ll T, \end{aligned}$$

as $T \rightarrow \infty$, for any fixed $\sigma > \frac{1}{2}$. Thus, condition (iii) holds also for any $\beta \in (\frac{1}{2}, 1)$. Since

$$\sum_{\substack{p \leq x \\ p \equiv h \pmod{\ell}}} |\sigma_\alpha(p)\chi(p)|^2 = \sum_{\substack{p \leq x \\ p \equiv h \pmod{\ell}}} 1 + O \left(\sum_{\substack{p \leq x \\ p \equiv h \pmod{\ell}}} p^a \right) = \pi(x; h, \ell) + O(1),$$

condition (iv) is also fulfilled. It remains to check condition (v). A short calculation shows

$$\sum_{j=1}^{\infty} \frac{|\sigma_{\alpha}(p^j)\chi(p^j)|}{p^{j\beta}} \leq \frac{1 + p^{a+\beta} - p^a - p^{\beta-a}}{(p^a - 1)(p^{\beta} - 1)(p^{\beta-a} - 1)}.$$

Therefore, if $a < -1$, then after an easy computation it turns out that inequality (8) holds for all primes $p \geq 5$ with any $\beta > \frac{1}{2}$. Hence, the assertion of the corollary follows from Theorem 5. •

Proof of Theorem 4. Let

$$c_{\chi} := \frac{\tau(\overline{\chi})\chi(k)}{\varphi(\ell)},$$

then $c_{\chi} \neq 0$. Then, by Theorem 2,

$$E\left(s; \frac{k}{\ell}, \alpha\right) = \sum_{\substack{\chi \bmod \ell \\ \chi \neq \chi_0}} c_{\chi} E(s; \chi, \alpha) + \Lambda(s; \alpha) E(s; \chi_0, \alpha). \quad (9)$$

Put $F := 1 + \sup_{s \in K} |g(s)|$. Then $g(s) + F \neq 0$ for $s \in K$. Now define

$$g_{\chi_1}(s) = \frac{g(s) + F}{c_{\chi_1}}, \quad g_{\chi_2} = -\frac{F}{c_{\chi_2}}, \quad \text{and} \quad g_{\chi_0}(s) = g_{\chi_j}(s) = \eta$$

for $j = 3, \dots, \ell - 2$, where η is a small positive parameter, chosen later; note that this is the point where we need $\ell \geq 5$. Hence, using (9), for any τ ,

$$\begin{aligned} E\left(s + i\tau, \frac{k}{\ell}, \alpha\right) - g(s) &= \sum_{\chi \neq \chi_0} c_{\chi} (E(s + i\tau; \chi, \alpha) - g_{\chi}(s)) + \eta \left(\sum_{j=3}^{\ell-1} c_{\chi_j} + \Lambda(s; \alpha) \right) \\ &\quad + \Lambda(s + i\tau; \alpha) E(s + i\tau; \chi_0, \alpha) - \Lambda(s; \alpha) g_{\chi_0}(s). \end{aligned}$$

Consequently,

$$\begin{aligned} &\max_{s \in K} \left| E\left(s + i\tau, \frac{k}{\ell}, \alpha\right) - g(s) \right| \\ &\leq \sum_{\chi \neq \chi_0} |c_{\chi}| \max_{s \in K} |E(s + i\tau, \chi, \alpha) - g_{\chi}(s)| + \eta \left(\sum_{j=3}^{\ell-2} |c_{\chi_j}| + \max_{s \in K} |\Lambda(s; \alpha)| \right) \\ &\quad + \max_{s \in K} |\Lambda(s + i\tau; \alpha) (E(s + i\tau, \chi_0, \alpha) - g_{\chi_0}(s))| \\ &\quad + \max_{s \in K} |\eta (\Lambda(s + i\tau; \alpha) - \Lambda(s; \alpha))|. \end{aligned} \quad (10)$$

By (5) it is easily seen that $\Lambda(s; \alpha) \ll \ell$ for $\frac{1}{2} < \sigma < 1$, and where the implicit constant does not depend on ℓ . Furthermore, Corollary 6 yields, for any $\delta > 0$,

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \max_{1 \leq j \leq \ell-2} \sup_{s \in K} |E(s + i\tau, \chi_j, \alpha) - g_{\chi_j}(s)| < \delta \right\} > 0.$$

Hence, any such τ gives in (10) the estimate

$$\max_{s \in K} \left| E\left(s + i\tau, \frac{k}{\ell}, \alpha\right) - g(s) \right| \leq C\ell(\delta + \eta),$$

where the constant C is absolute. Taking δ and η sufficiently small, the theorem is shown. •

It remains to show that the estimate (8) holds for $p = 2, 3$ with certain values of α in the half plane $a < -1$ and some $\beta \in (\frac{1}{2}, 1)$. Therefore, take $\beta = \frac{4}{5}$, then (8) is fulfilled for $p = 3$. Further, choosing α with

$$\operatorname{Im} \alpha \equiv \frac{\pi}{\log 2} \pmod{2\pi},$$

leads to

$$\sigma_\alpha(2^j) = 1 - 2^a \pm \dots + (-1)^j 2^a = \frac{1 - (-2^a)^{j+1}}{1 + 2^a},$$

and thus the left hand side of (8) equals

$$\sum_{j=1}^{\infty} \frac{|\sigma_\alpha(2^j)|}{2^{j0.8}} = \frac{1 - 2^{a+0.8} + 2^a + 2^{0.8-a}}{(2^a + 1)(2^{0.8} - 1)(2^{0.8-a} + 1)},$$

which is < 1 for $-1.7 \leq a < -1$. Hence, for example $E(s; \frac{1}{5}, -\frac{3}{2} + i\frac{\pi}{\log 2})$ has the universality property in the strip $\frac{4}{5} < \sigma < 1$. Obviously, a choice of β close to 1 implies universality for $E(s; \frac{k}{\ell}, \alpha)$ with less restricted α . Of course, the restrictions on ℓ and α can be easily relaxed but we cannot remove restriction (8) in Theorem 4.

We have failed to prove universality for arbitrary Estermann zeta-functions $E(s; \frac{k}{\ell}, \alpha)$ or functions $E(s; \chi, \alpha)$. However, we believe that all in this paper appearing zeta-functions have the universality property.

4 Applications

We conclude with some immediate consequences of the universality property. Therefore, we assume in what follows that $E(s; \frac{k}{\ell}, \alpha)$ satisfies all conditions of Theorem 4.

As in the case of the Riemann zeta-function one can show that, for fixed $\sigma \in (\frac{1}{2}, 1)$, the set

$$\left\{ \left(E\left(\sigma + it; \frac{k}{\ell}, \alpha\right), E'\left(\sigma + it; \frac{k}{\ell}, \alpha\right), \dots, E^{(n-1)}\left(\sigma + it; \frac{k}{\ell}, \alpha\right) \right) : t \in \mathbb{R} \right\}$$

lies everywhere dense in $\mathbb{C}^n$.

Further one can prove functional independence: if $F_1(z_1, \dots, z_n), \dots, F_N(z_1, \dots, z_n)$ are continuous functions on $\mathbb{C}^n$, not all identically zero, then there exists some $s \in \mathbb{C}$ for which

$$\sum_{j=1}^N s^j F_j \left(E \left(s; \frac{k}{\ell}, \alpha \right), E' \left(s; \frac{k}{\ell}, \alpha \right), \dots, E^{(n-1)} \left(s; \frac{k}{\ell}, \alpha \right) \right) \neq 0.$$

Finally, Rouché's theorem implies that for any σ_2, σ_3 satisfying $\frac{1}{2} < \sigma_2 < \sigma_3 < 1$, and for sufficiently large T , there exist at least cT zeros of $E(s; \frac{k}{\ell}, \alpha)$ in the region $\sigma_2 < \sigma < \sigma_3, |t| \leq T$, where c is a positive constant independent on T . In particular, the real parts of the zeros lie dense in $(\frac{1}{2}, 1)$. In combination with the upper estimates of [12] we obtain now for the number $N(\sigma, T; \frac{k}{\ell}, \alpha)$ of zeros $\rho = \beta + i\gamma$ of $E(s; \frac{k}{\ell}, \alpha)$ with $\beta > \sigma$ and $|\gamma| \leq T$ (according multiplicities) the right asymptotic order.

Corollary 7 *Under the assumptions of Theorem 4, we have for fixed $\sigma \in (\frac{1}{2}, 1)$, as $T \rightarrow \infty$,*

$$T \ll N \left(\sigma, T; \frac{k}{\ell}, \alpha \right) \ll T.$$

Therefore, Estermann zeta-functions with the universality property of Theorem 4 have many zeros off the lines $\sigma = \frac{1}{2}$ and $\sigma = \frac{1}{2} + a$ whereas for $E(s; 1, \alpha) = \zeta(s)\zeta(s - \alpha)$ it is expected that all nontrivial zeros (complex if $\alpha \in \mathbb{R}$) lie exactly on these lines.

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References

- [1] T.M. APOSTOL, *Introduction to Analytic Number Theory*, Springer 1976
- [2] B. BAGCHI, *A joint universality theorem for Dirichlet L-functions*, Math. Z. **181** (1992), 319-334

- [3] T. ESTERMANN, *On the representation of a number as the sum of two products*, Proc. London Math. Soc. **31** (1930), 123-133
- [4] S.M. GONEK, *Analytic properties of zeta and L-functions*, Ph.D.Thesis, University of Michigan, 1979
- [5] I. KIUCHI, *On an exponential sum involving the arithmetic function $\sigma_a(n)$* , Math. J. Okayama Univ. **29** (1987), 193-205
- [6] A. LAURINČIKAS, *Limit theorems for the Riemann zeta-function*, Kluwer Academic Publishers, Dordrecht 1996
- [7] A. LAURINČIKAS, W. SCHWARZ, J. STEUDING, *The universality of general Dirichlet series*, preprint
- [8] H. NAKAYA, *The mean square of the Dirichlet L-functions in the critical strip*, Liet. matem. rink. **40** (2000), 201-213
- [9] A. REICH, *Universelle Wertverteilung von Eulerprodukten*, Nach. Akad. Wiss. Göttingen, Math.-Phys. Kl. (1977), 1-17
- [10] K.F. ROTH, R.C. VAUGHAN, *Theodor Estermann - Obituary*, Bull. London Math. Soc. **26** (1994), 593-606
- [11] R. ŠLEŽEVIČIENĖ, *The joint universality for twists of Dirichlet series with multiplicative coefficients by characters*, submitted for publication
- [12] R. ŠLEŽEVIČIENĖ, J. STEUDING, *On the zeros of the Estermann zeta-function*, Integral transforms and special functions (to appear)
- [13] E. C. TITCHMARSH, *The Theory of the Riemann Zeta-Function*, Oxford: The Clarendon Press (1986)
- [14] S.M. VORONIN, *Theorem on the 'universality' of the Riemann zeta-function*, Izv. Akad. Nauk SSSR, Ser. Matem., **39** (1975) (in Russian); Math. USSR Izv. **9** (1975), 443-445