THE DISCRETE MEAN SQUARE OF THE DIRICHLET L -FUNCTION AT NONTRIVIAL ZEROS OF ANOTHER DIRICHLET L -FUNCTION

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We consider the sum of squared absolute values of the Dirichlet L -function taken at the nontrivial zeros of another Dirichlet L -function.

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1. Introduction

Let $s = \sigma + it$ denote a complex variable. The Dirichlet L -function is defined by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \quad (\sigma > 1),$$

where $\chi(n)$ is a Dirichlet character modulo q . For $\chi \bmod 1$ we get the Riemann zeta-function $L(s, \chi) = \zeta(s)$. The Generalized Riemann Hypothesis (GRH) states that inside the critical strip $0 < \sigma < 1$ every Dirichlet L -function has zeros only on the critical line $\sigma = \frac{1}{2}$. Zeros in the critical strip are called nontrivial and we denote them by $\rho_\chi = \beta_\chi + i\gamma_\chi$. A Dirichlet character $\chi \bmod q$ is said to be primitive if it is not induced by any other character of modulus strictly less than q . The unique principal character modulo q is denoted by χ_0 . The character $\chi_0 \bmod 1$ is the only principal and primitive character. For a Dirichlet character $\chi \bmod q$ the associated Gauss sum is defined by

$$G(n, \chi) = \sum_{a=1}^q \chi(a) \exp\left(2\pi i \frac{an}{q}\right).$$

If $n = 1$ we denote $\tau(\chi) = G(1, \chi)$. For a primitive character $\chi \bmod q$ we have $|\tau(\chi)| = \sqrt{q}$ and for the principal character χ_0 we have $\tau(\chi_0) = \mu(q)$, where $\mu(q)$ is the Möbius function.

In this paper T always tends to plus infinity, ε is any positive real number, and p denotes a prime number. Let ϕ denote the Euler totient function and let

$$D(Q) = 2 \sum_{p|Q} \frac{\log p}{p-1} + 2\gamma,$$

where γ is the Euler constant. Further let $\delta(q, Q) = 1$, if $q|Q$, $\delta(q, Q) = 0$ otherwise. We shall prove the following result.

Theorem 1.1. *Let A and B be positive constants. Let $\psi \bmod Q$ and $\chi \bmod q$ be primitive Dirichlet characters and $\chi \neq \psi$. Then, uniformly for $Q \ll \log^A T$ and $q \ll \log^B T$, we have*

$$\begin{aligned} \sum_{0 < \gamma_\chi \leq T} L(\rho_\chi, \psi) L(1 - \rho_\chi, \bar{\psi}) &= \frac{\phi(Q)}{Q} \frac{T}{2\pi} \log^2 \frac{T}{2\pi e} + a_1 \frac{T}{2\pi} \log \frac{T}{2\pi e} + a_2 \frac{T}{2\pi} \\ &\quad + O\left(T^{1 - \frac{c}{\log^{\frac{3}{4} + \varepsilon} T}}\right), \end{aligned} \quad (1.1)$$

where

$$a_1 = \frac{\phi(Q)}{Q} \left(\log(Qq) + D(Q) + 2\Re \frac{L'}{L}(1, \chi\bar{\psi}) \right)$$

and

$$\begin{aligned} a_2 &= \frac{\phi(Q)}{Q} \left[\log Q \log q + D(Q) \log q + 1 + 2(\log Q + D(Q)) \Re \frac{L'}{L}(1, \chi\bar{\psi}) \right. \\ &\quad \left. + 2\Re \left(\frac{L''}{L} - \left(\frac{L'}{L} \right)^2 \right) (1, \chi\bar{\psi}) \right] - 2\delta(q, Q)(\chi\bar{\psi})(-1) \\ &\quad \times \Re \left(\frac{\tau(\chi\psi_0)\tau(\bar{\psi})}{\phi(Q)} L^2(1, \chi\bar{\psi}) \right). \end{aligned}$$

If we assume GRH then the left-hand side of the last equality can be replaced by $\sum_{0 < \gamma_\chi \leq T} |L(\frac{1}{2} + i\gamma_\chi, \psi)|^2$ and the error term can be replaced by $O(q^{1+\varepsilon} Q^\varepsilon T^{\frac{1}{2}+\varepsilon} + qQ^{\frac{9}{2}+\varepsilon} T^\varepsilon + (QT)^{\frac{1}{2}+\varepsilon})$ uniformly for all Q and q .

From Theorem 1.1 we immediately deduce the following corollary.

Corollary 1.2. *Let $\psi \bmod Q$ and $\chi \bmod q$ be primitive Dirichlet characters and $\chi \neq \psi$. Assume that all nontrivial zeros of $L(s, \chi)$ lie on the critical line. Then the sequence $(L(\rho_\chi, \psi))$, where ρ_χ runs over all nontrivial zeros of $L(s, \chi)$, is unbounded.*

Theorem 1.1 is related to the following theorem proved in [11].

Theorem 1.3. *Let A and B be positive constants. Let $\psi \bmod Q$ and $\chi \bmod q$ be primitive Dirichlet characters and $\chi \neq \psi$. Then, uniformly for $Q \ll \log^A T$ and $q \ll \log^B T$, we have*

$$\sum_{0 < \gamma_\chi \leq T} L(\rho_\chi, \psi) = \frac{T}{2\pi} \log \frac{Tq}{2\pi e} - \delta(q, Q) L(1, \chi \overline{\psi}) \psi(-1) \tau(\psi) \frac{\tau(\overline{\chi} \psi_0)}{\phi(Q)} \frac{T}{2\pi} \\ + \frac{L'}{L}(1, \psi \overline{\chi}) \frac{T}{2\pi} + O(T \exp(-c \log^{\frac{1}{4}-\varepsilon} T)),$$

where c is a positive absolute constant.

Under GRH the error term can be replaced by $O((TQ)^{1/2+\varepsilon} q^\varepsilon)$, which is valid uniformly for $q, Q \ll T^{1-\varepsilon}$.

Assuming GRH for $L(s, \chi)$ Theorem 1.3 was proved by Fujii [8] with the error term $O(T \exp(-c \log^{\frac{1}{2}} T))$. Later Steuding [19] proved Theorem 1.3 unconditionally with the error term $O(T)$. See also [14], where the case $Q = 1$ and $q \leq \exp(c \sqrt{\log \frac{T}{2\pi}})$, c being an appropriate positive constant, was considered.

Theorems 1.1 and 1.3 lead to the next corollary.

Corollary 1.4. *Let $\psi \bmod Q$ and $\chi \bmod q$ be primitive Dirichlet characters and $\chi \neq \psi$. Assume that all nontrivial zeros of $L(s, \chi)$ lie on the critical line. Let A be any positive real number. Then, uniformly for $q \ll (\log T)^A$ and $Q \ll (\log T)^{2-\varepsilon}$, we have*

$$\sum_{\substack{0 < \gamma_\chi \leq T \\ L(1/2 + \gamma_\chi, \psi) \neq 0}} 1 \gg \frac{Q}{\phi(Q)} T.$$

The above results give certain information concerning the relationship of zeros of two Dirichlet L -functions. The conjecture is that different primitive Dirichlet L -functions have no common nontrivial zeros (see [9, Conjecture 3; 16]). Towards this hypothesis Fujii [7] unconditionally obtained, that if χ_1 and χ_2 are different primitive characters to the same modulus q , then the positive proportion of zeros of $L(s, \chi_1)$ and $L(s, \chi_2)$ are non-coincident. Here a zero ρ is called a coincident zero of $L(s, \chi_1)$ and $L(s, \chi_2)$ if $L(\rho, \chi_1) = L(\rho, \chi_2) = 0$ with the same multiplicity. Assuming the Riemann Hypothesis for $\zeta(s)$ Conrey, Ghosh, and Gonek [2, 3] proved that at most two-thirds of zeros of $\zeta(s)$ are also zeros of $L(s, \chi)$, where χ is a non-principle character. Conrey, Ghosh, and Gonek [2, 3] also note that their method allows to show (under GRH) that any two Dirichlet L -functions with inequivalent characters have at most two-thirds of zeros in common. For related results see also [18].

In the following section we will recall some basic facts about the analytic properties of Dirichlet L -functions and will give several lemmas. In Secs. 3 and 4 we will prove Theorem 1.1 and Corollary 1.4, respectively.

2. Auxiliary Results

First we recall several facts about Dirichlet L -functions. Dirichlet L -function to a primitive character $\psi \bmod Q$ satisfies the functional equation (see [1, Theorem 12.11])

$$L(1-s, \psi) = \tau(\psi) \frac{1}{Q} \left(\frac{Q}{2\pi} \right)^s \Gamma(s) \left(\exp\left(-\frac{\pi i s}{2}\right) + \psi(-1) \exp\left(\frac{\pi i s}{2}\right) \right) L(s, \overline{\psi}). \quad (2.1)$$

Thus theorem from [13] and an application of the Phragmén–Lindelöf principle yield the estimates

$$L(s, \psi) \ll |QT|^{\frac{3}{16}+\varepsilon} \quad \text{for } \frac{1}{2} \leq \sigma \leq 1 + \frac{1}{\log(QT)}, \quad |t| \geq 1, \quad (2.2)$$

$$L(s, \psi) \ll |QT|^{\frac{1}{2}+\varepsilon} \quad \text{for } -\frac{1}{\log(QT)} \leq \sigma < \frac{1}{2}, \quad |t| \geq 1 \quad (2.3)$$

uniformly in $|t| \ll T$. Under GRH the bound (2.2) can be replaced by

$$L(s, \psi) \ll |QT|^\varepsilon \quad \text{for } \frac{1}{2} \leq \sigma \leq 1 + \frac{1}{\log(QT)}, \quad |t| \geq 1 \quad (2.4)$$

uniformly in $|t| \ll T$. The bound (2.4) can be obtained similarly as in [20, Theorem 13.5], see also [10, Theorem 4].

We rewrite the functional equation (2.1) as

$$L(1-s, \psi) = \Delta(1-s, \psi) L(s, \overline{\psi}), \quad (2.5)$$

where $\Delta(s, \psi)$ is a meromorphic function with only real zeros and poles satisfying the functional equation

$$\Delta(s, \psi) \Delta(1-s, \overline{\psi}) = 1. \quad (2.6)$$

Stirling's formula gives, for $t > 1$,

$$\Delta(\sigma + it, \psi) = \left(\frac{tQ}{2\pi} \right)^{\sigma - \frac{1}{2} - it} \exp\left(i \left(t + \frac{\pi}{4} \right)\right) \left(1 + O\left(\frac{1}{t}\right) \right) \quad (2.7)$$

and

$$\frac{\Delta'}{\Delta}(s, \psi) = \frac{\Delta'}{\Delta}(1-s, \overline{\psi}) = -\log \frac{tQ}{2\pi} + O\left(\frac{1}{t}\right). \quad (2.8)$$

Logarithmic differentiation of (2.5) leads to

$$\frac{L'}{L}(1-s, \chi) = \frac{\Delta'}{\Delta}(1-s, \chi) - \frac{L'}{L}(s, \overline{\chi}). \quad (2.9)$$

For the logarithmic derivative we have the partial fraction decomposition (see [17, Chap. 7, Theorem 4.1])

$$\frac{L'}{L}(s, \chi) = \sum_{|t-\gamma_\chi| \leq 1} \frac{1}{s - \rho_\chi} + O(\log(q(|t| + 2))) \quad \text{for } -1 \leq \sigma \leq 2, \quad t \geq 1. \quad (2.10)$$

For $q \geq 1$, $\chi \bmod q$ and $t \geq 0$ we have (see [17, Chap. 7, Theorem 3.3])

$$\begin{aligned} N_\chi(t+1) - N_\chi(t) &:= \#\{\rho_\chi = \beta_\chi + i\gamma_\chi : t < \gamma_\chi \leq t+1\} \\ &\ll \log(q(t+2)). \end{aligned} \quad (2.11)$$

Thus the zeros ρ_χ cannot lie too dense: for any given $t_0 \geq 2$ there exists a $t = t(\chi)$, $t \in (t_0, t_0 + 1]$, such that

$$\min_{\gamma_\chi} |t - \gamma_\chi| \gg \frac{1}{\log(qt)}. \quad (2.12)$$

In view of the expression (2.10) we get, for t satisfying (2.12),

$$\frac{L'}{L}(\sigma + it, \chi) \ll \log^2(qt) \quad \text{for } -1 \leq \sigma \leq 2, \quad t \geq 2. \quad (2.13)$$

In the proof of Theorem 1.1 the following modified Gonek lemma (cf. [12, Lemma 5] and [5, Lemma 2]) will be important.

Lemma 2.1. *Assume that $\sum_{n=1}^{\infty} \frac{a(n)}{n^\sigma}$ converges for $\sigma > 1$ and $a(n) = O(n^\varepsilon)$. Let $a = 1 + \frac{1}{\log(QT)}$. Then*

$$\begin{aligned} &\frac{1}{2\pi i} \int_{a-i}^{a+iT} \left(\frac{m}{2\pi}\right)^s \Gamma(s) \exp\left(\delta \frac{\pi i s}{2}\right) \sum_{n=1}^{\infty} \frac{a(n)}{n^s} ds \\ &= \begin{cases} \sum_{n \leq \frac{Tm}{2\pi}} a(n) \exp\left(-2\pi i \frac{n}{m}\right) + O(m^a T^{1/2+\varepsilon}) & \text{if } \delta = -1, \\ O(m^a) & \text{if } \delta = +1. \end{cases} \end{aligned}$$

Proof. This is Lemma 2 in [11]. □

The next two technical lemmas will be useful in the proof of Theorem 1.1.

Lemma 2.2. *Let $\chi \bmod q$ and $\psi \bmod Q$ be primitive Dirichlet characters and $\chi \neq \psi$ with $Q \ll \log^A T$ and $q \ll \log^B T$, where A and B are positive constants. We have*

$$\begin{aligned} &\sum_{mnl \leq \frac{TQ}{2\pi}} \Lambda(m) \chi(m) \psi(n) \psi(l) \exp\left(-2\pi i \frac{mnl}{Q}\right) \\ &= -\psi(-1) \tau(\psi) \frac{\phi(Q)}{Q} \frac{L'}{L}(1, \chi \bar{\psi}) \frac{T}{2\pi} \left(\log \frac{TQ}{2\pi e} + 2\gamma + 2 \sum_{p|Q} \frac{\log p}{p-1} \right. \\ &\quad \left. + \left(\frac{L''}{L'} - \frac{L'}{L} \right) (1, \chi \bar{\psi}) \right) + \delta(q, Q) \chi(-1) \tau(\chi \psi_0) \frac{1}{\phi(Q)} \frac{TQ}{2\pi} L^2(1, \chi \bar{\psi}) \\ &\quad + O\left(T^{1 - \frac{c}{\log \frac{3}{4} + \varepsilon} T}\right). \end{aligned}$$

Under GRH the error term can be replaced by

$$O(q Q(TQ)^{\frac{1}{2}+\varepsilon} \log^2 q + q Q^5 \log(QT))$$

uniformly for all Q and q .

Proof. By the orthogonality of Dirichlet characters the left-hand side of the formula in Lemma 2.2 can be written in the following way:

$$\begin{aligned} & \sum_{mnl \leq \frac{TQ}{2\pi}} \Lambda(m) \chi(m) \psi(n) \psi(l) \exp\left(-2\pi i \frac{mnl}{Q}\right) \\ &= \sum_{a=1}^{[q,Q]} \chi(a) \sum_{b=1}^Q \psi(b) \sum_{c=1}^Q \psi(c) \exp\left(-2\pi i \frac{abc}{Q}\right) \sum_{\substack{mnl \leq \frac{TQ}{2\pi} \\ m \equiv a \pmod{[q,Q]} \\ n \equiv b \pmod{Q} \\ l \equiv c \pmod{Q}}} \Lambda(m) \\ &= \frac{1}{\phi^2(Q)} \sum_{\substack{\omega' \pmod{Q} \\ \omega'' \pmod{Q}}} \sum_{a=1}^{[q,Q]} \chi(a) \sum_{b=1}^Q \overline{\omega'}(b) \psi(b) \sum_{c=1}^Q \overline{\omega''}(c) \psi(c) \exp\left(-2\pi i \frac{abc}{Q}\right) \\ &\quad \times \sum_{\substack{mnl \leq \frac{TQ}{2\pi} \\ m \equiv a \pmod{[q,Q]}}} \Lambda(m) \omega'(n) \omega''(l) = S_1 + S_2 + S_3 + S_4, \end{aligned} \quad (2.14)$$

where S_j , $j = 1, 2, 3, 4$ correspond to the following cases:

- S_1 : $\omega' = \omega'_0$ and $\omega'' = \omega''_0$,
- S_2 : $\omega' = \omega'_0$ and $\omega'' \neq \omega''_0$,
- S_3 : $\omega' \neq \omega'_0$ and $\omega'' = \omega''_0$,
- S_4 : $\omega' \neq \omega'_0$ and $\omega'' \neq \omega''_0$,

here $\omega'_0 = \omega''_0$ is the principal character modulo Q .

First we consider S_1 . Since ψ is a primitive Dirichlet character, the Gauss sum is separable, i.e. $G(-ab, \psi) = \overline{\psi}(-ab) \tau(\psi)$ (see [6, Sec. 9]). By this and by the orthogonality of Dirichlet characters we have

$$\begin{aligned} S_1 &= \psi(-1) \tau(\psi) \frac{1}{\phi(Q)} \sum_{a=1}^{[q,Q]} \chi(a) \overline{\psi}(a) \sum_{\substack{mnl \leq \frac{TQ}{2\pi} \\ m \equiv a \pmod{[q,Q]}}} \Lambda(m) \omega'_0(n) \omega''_0(l) \\ &= \psi(-1) \tau(\psi) \frac{1}{\phi([q,Q])} \frac{1}{\phi(Q)} \sum_{\eta \pmod{[q,Q]}} \sum_{a=1}^{[q,Q]} \overline{\eta}(a) \chi(a) \overline{\psi}(a) \\ &\quad \times \sum_{mnl \leq \frac{TQ}{2\pi}} \Lambda(m) \eta(m) \omega'_0(n) \omega''_0(l). \end{aligned}$$

In the last formula the sum over a is not equal to zero if and only if $\eta = \chi\bar{\psi}$. Thus

$$S_1 = \psi(-1)\tau(\psi)\frac{1}{\phi(Q)} \sum_{mnl \leq \frac{TQ}{2\pi}} \Lambda(m)(\chi\bar{\psi})(m)\omega'_0(n)\omega''_0(l). \quad (2.15)$$

Second we consider S_2 . In formula (2.14) interchanging the summation (over b and c) and using the equality $G(-ac, \psi) = \bar{\psi}(-ac)\tau(\psi)$ we obtain

$$\begin{aligned} S_2 &= \frac{1}{\phi^2(Q)} \sum_{\substack{\omega'' \bmod Q \\ \omega'' \neq \omega'_0}} \sum_{a=1}^{[q, Q]} \chi(a) \sum_{c=1}^Q \bar{\omega}''(c) \psi(c) \sum_{b=1}^Q \psi(b) \exp\left(-2\pi i \frac{abc}{Q}\right) \\ &\quad \times \sum_{\substack{mnl \leq \frac{TQ}{2\pi} \\ m \equiv a \bmod [q, Q]}} \Lambda(m) \omega'_0(n) \omega''(l) \\ &= \psi(-1)\tau(\psi) \frac{1}{\phi^2(Q)} \sum_{\substack{\omega' \bmod Q \\ \omega' \neq \omega'_0}} \sum_{a=1}^{[q, Q]} \chi(a) \bar{\psi}(a) \sum_{c=1}^Q \bar{\omega}''(c) \\ &\quad \times \sum_{\substack{mnl \leq \frac{TQ}{2\pi} \\ m \equiv a \bmod [q, Q]}} \Lambda(m) \omega'_0(n) \omega''(l). \end{aligned}$$

Hence $S_2 = 0$, since the sum over c is equal to zero.

In the same way we obtain $S_3 = 0$.

Finally we consider S_4 . First we note that, for $\omega' \neq \omega'_0$, $\omega'' \neq \omega''_0$ and $(a, [q, Q]) > 1$,

$$\begin{aligned} \left| \sum_{\substack{m \leq \frac{TQ}{2\pi} \\ m \equiv a \bmod [q, Q]}} \Lambda(m) \sum_{\substack{nl \leq \frac{TQ}{2\pi m}}} \omega'(n) \omega''(l) \right| &\leq \phi^2(Q) \sum_{\substack{m \leq \frac{TQ}{2\pi} \\ m \equiv a \bmod [q, Q]}} \Lambda(m) \\ &= O(\phi^2(Q) \log(TQ)). \end{aligned}$$

This yields

$$\begin{aligned} S_4 &= \frac{1}{\phi^2(Q)} \sum_{\substack{\omega' \bmod Q \\ \omega' \neq \omega'_0}} \sum_{\substack{\omega'' \bmod Q \\ \omega'' \neq \omega''_0}} \sum_{a=1}^{[q, Q]} \chi(a) \sum_{b=1}^Q \bar{\omega}'(b) \psi(b) \\ &\quad \times \sum_{c=1}^Q \bar{\omega}''(c) \psi(c) \exp\left(-2\pi i \frac{abc}{Q}\right) \\ &\quad \times \sum_{\substack{mnl \leq \frac{TQ}{2\pi} \\ m \equiv a \bmod [q, Q] \\ (a, [q, Q])=1}} \Lambda(m) \omega'(n) \omega''(l) + O(\phi^4(Q) \phi([q, Q]) \log(TQ)). \end{aligned}$$

If $(ab, Q) = 1$, then $G(-ab, \overline{\omega''}\psi) = (\omega''\overline{\psi})(-ab)\tau(\overline{\omega''}\psi)$ (see [6, Sec. 9]). Then by the orthogonality of Dirichlet characters we have

$$\begin{aligned} S_4 &= \frac{1}{\phi([q, Q])} \frac{1}{\phi^2(Q)} \sum_{\eta \bmod [q, Q]} \sum_{\substack{\omega' \bmod Q \\ \omega' \neq \omega'_0}} \sum_{\substack{\omega'' \bmod Q \\ \omega'' \neq \omega''_0}} (\omega''\overline{\psi})(-1)\tau(\overline{\omega''}\psi) \\ &\quad \times \sum_{a=1}^{[q, Q]} \overline{\eta}(a)\chi(a)\omega''(a)\overline{\psi}(a) \sum_{b=1}^Q \overline{\omega'}(b)\omega''(b) \sum_{mnl \leq \frac{TQ}{2\pi}} \Lambda(m)\eta(m)\omega'(n)\omega''(l) \\ &\quad + O(\phi^4(Q)\phi([q, Q])\log(TQ)). \end{aligned}$$

In the last formula the sum over b does not vanish if and only if $\omega' = \omega''$. Further we write $S_4 = T_1 + T_2$, where T_1 and T_2 correspond to the following cases:

- $T_1: \eta = \eta_0$,
- $T_2: \eta \neq \eta_0$.

Let $\delta(q, Q) = 1$ if $q|Q$ and $\delta(q, Q) = 0$ otherwise. Then

$$\begin{aligned} T_1 &= \frac{1}{\phi([q, Q])} \frac{1}{\phi(Q)} \sum_{\substack{\omega' \bmod Q \\ \omega' \neq \omega'_0}} (\omega'\overline{\psi})(-1)\tau(\overline{\omega'}\psi) \sum_{a=1}^{[q, Q]} \eta_0(a)\chi(a)\omega'(a)\overline{\psi}(a) \\ &\quad \times \sum_{mnl \leq \frac{TQ}{2\pi}} \Lambda(m)\eta_0(m)\omega'(n)\omega''(l) \\ &= \delta(q, Q)(\chi\psi_0)(-1)\tau(\chi\psi_0) \frac{1}{\phi(Q)} \\ &\quad \times \sum_{mnl \leq \frac{TQ}{2\pi}} \Lambda(m)\eta_0(m)(\overline{\chi}\psi)(n)(\overline{\chi}\psi)(l), \end{aligned} \tag{2.16}$$

since the sum over a does not vanish if and only if $\omega' = \overline{\chi}\psi$; and this equality is true if and only if $q|Q$. By definition of T_2 we have

$$\begin{aligned} T_2 &= \frac{1}{\phi([q, Q])} \frac{1}{\phi(Q)} \sum_{\substack{\eta \bmod [q, Q] \\ \eta \neq \eta_0}} \sum_{\substack{\omega' \bmod Q \\ \omega' \neq \omega'_0}} (\omega'\overline{\psi})(-1)\tau(\overline{\omega'}\psi) \\ &\quad \times \sum_{a=1}^{[q, Q]} \overline{\eta}(a)\chi(a)\omega'(a)\overline{\psi}(a) \sum_{mnl \leq \frac{TQ}{2\pi}} \Lambda(m)\eta(m)\omega'(n)\omega''(l), \end{aligned} \tag{2.17}$$

where ω'' is a non-principal character mod Q .

In view of the above we can write

$$\begin{aligned} &\sum_{mnl \leq \frac{TQ}{2\pi}} \Lambda(m)\chi(m)\psi(n)\psi(l) \exp\left(-2\pi i \frac{mnl}{Q}\right) \\ &= S_1 + T_1 + T_2 O(\phi^4(Q)\phi([q, Q])\log(TQ)), \end{aligned}$$

where S_1 , T_1 , and T_2 are given by formulae (2.15), (2.16), and (2.17). We will see that sums S_1 and T_1 produce the main term, and the sum T_2 contributes to the error term.

Next we consider S_1 . Note that by conditions of the theorem the character $\chi\bar{\psi}$ is a non-principal character. By Perron's formula (see [20, Lemma 3.12])

$$\begin{aligned} -\frac{\phi(Q)}{\psi(-1)\tau(\psi)}S_1 &= -\sum_{mnl \leq \frac{TQ}{2\pi}} \Lambda(m)(\chi\bar{\psi})(m)\omega'_0(n)\omega''_0(l) \\ &= \frac{1}{2\pi i} \int_{a-iU}^{a+iU} \frac{L'}{L}(s, \chi\bar{\psi})L(s, \omega'_0)L(s, \omega''_0) \left(\frac{TQ}{2\pi}\right)^s \frac{ds}{s} \\ &\quad + O\left(\frac{TQ \log^2(TQ)}{U}\right), \end{aligned} \quad (2.18)$$

where $a = 1 + 1/\log(TQ)$.

In [17, Chap. 8, Theorem 6.2] considering $q \ll \log^B T$ with B being any positive constant we find that

$$L(s, \chi) \neq 0 \quad \text{for } \sigma > 1 - \frac{c}{\log^{\frac{3}{4}+\varepsilon} T},$$

where c is an absolute positive constant. With regard to this zero-free region for $L(s, \chi)$ let $b_1 = 1 - c/\log^{\frac{3}{4}+\varepsilon} T$. Shifting the line of integration and noting that $L(s, \omega'_0) = L(s, \omega''_0)$ we get

$$\begin{aligned} &-\sum_{mnl \leq \frac{TQ}{2\pi}} \Lambda(m)(\chi\bar{\psi})(m)\omega'_0(n)\omega''_0(l) \\ &= \operatorname{Res}_{s=1} \frac{L'}{L}(s, \chi\bar{\psi})L^2(s, \omega'_0) \left(\frac{TQ}{2\pi}\right)^s \frac{1}{s} \\ &\quad - \frac{1}{2\pi i} \left\{ \int_{a+iU}^{b_1+iU} + \int_{b_1+iU}^{b_1-iU} + \int_{b_1-iU}^{a-iU} \right\} \frac{L'}{L}(s, \chi\bar{\psi})L^2(s, \omega'_0) \left(\frac{TQ}{2\pi}\right)^s \frac{ds}{s} \\ &\quad + O\left(\frac{TQ \log^2(TQ)}{U}\right). \end{aligned} \quad (2.19)$$

Note that

$$\begin{aligned} &\operatorname{Res}_{s=1} \frac{L'}{L}(s, \chi\bar{\psi})L^2(s, \omega'_0) \left(\frac{TQ}{2\pi}\right)^s \frac{1}{s} \\ &= \lim_{s \rightarrow 1} \frac{d}{ds} \left((s-1)^2 \frac{L'}{L}(s, \chi\bar{\psi})L^2(s, \omega'_0) \left(\frac{TQ}{2\pi}\right)^s \frac{1}{s} \right). \end{aligned}$$

To calculate this residue we use the following expansion (see [6, Sec. 4, formula (6); 20, formula (2.1.16)])

$$\begin{aligned} L^2(s, \omega'_0) &= \left(\zeta(s) \prod_{p|Q} (1 - p^{-s}) \right)^2 \\ &= \left(\frac{1}{s-1} + \gamma + \sum_{n=1}^{\infty} \gamma_n (s-1)^n \right)^2 \prod_{p|Q} (1 - p^{-s})^2 \\ &= \left(\frac{1}{(s-1)^2} + \frac{2\gamma}{(s-1)} + \sum_{m=0}^{\infty} a_m (s-1)^m \right) \prod_{p|Q} (1 - p^{-s})^2, \end{aligned}$$

where γ is the Euler constant, γ_n , $n = 1, 2, \dots$, and a_m , $m = 0, 1, \dots$ are absolute real constants. Then by (see [1, Theorem 2.4])

$$\prod_{p|Q} (1 - p^{-1}) = \frac{\phi(Q)}{Q}, \quad (2.20)$$

$$\frac{d}{ds} \left(\prod_{p|Q} (1 - p^{-s}) \right)^2 = 2 \prod_{p|Q} (1 - p^{-s})^2 \sum_{p|Q} \frac{\log p}{p^s - 1}, \quad (2.21)$$

and

$$\frac{d}{ds} \frac{L'}{L}(s, \chi\bar{\psi}) = \left(\frac{L''L - (L')^2}{L^2} \right) (s, \chi\bar{\psi})$$

we get

$$\begin{aligned} \operatorname{Res}_{s=1} \frac{L'}{L}(s, \chi\bar{\psi}) L^2(s, \omega'_0) &\left(\frac{TQ}{2\pi} \right)^s \frac{1}{s} \\ &= \lim_{s \rightarrow 1} \frac{d}{ds} \left((1 + 2\gamma(s-1)) \frac{L'}{L}(s, \chi\bar{\psi}) \left(\frac{TQ}{2\pi} \right)^s \frac{1}{s} \prod_{p|Q} (1 - p^{-s})^2 \right) \\ &= (2\gamma - 1) \frac{L'}{L}(1, \chi\bar{\psi}) \frac{TQ}{2\pi} \prod_{p|Q} (1 - p^{-1})^2 + \frac{L''L - (L')^2}{L^2}(1, \chi\bar{\psi}) \frac{TQ}{2\pi} \prod_{p|Q} (1 - p^{-1})^2 \\ &\quad + \prod_{p|Q} (1 - p^{-1})^2 \frac{L'}{L}(1, \chi\bar{\psi}) \frac{TQ}{2\pi} \log \frac{TQ}{2\pi} + 2 \frac{L'}{L}(1, \chi\bar{\psi}) \frac{TQ}{2\pi} \\ &\quad \times \prod_{p|Q} (1 - p^{-1})^2 \sum_{p|Q} \frac{\log p}{p-1} \\ &= \left(\frac{\phi(Q)}{Q} \right)^2 \frac{L'}{L}(1, \chi\bar{\psi}) \frac{TQ}{2\pi} \left(\log \frac{TQ}{2\pi e} + 2\gamma + 2 \sum_{p|Q} \frac{\log p}{p-1} + \left(\frac{L''}{L'} - \frac{L'}{L} \right) (1, \chi\bar{\psi}) \right). \end{aligned}$$

This and formulae (2.19), (2.18) yield

$$S_1 = -\psi(-1)\tau(\psi)\frac{\phi(Q)}{Q}\frac{L'}{L}(1, \chi\bar{\psi})\frac{T}{2\pi}\left(\log\frac{TQ}{2\pi e} + 2\gamma + 2\sum_{p|Q}\frac{\log p}{p-1}\right. \\ \left. + \left(\frac{L''}{L'} - \frac{L'}{L}\right)(1, \chi\bar{\psi})\right) + \mathcal{E},$$

where

$$\mathcal{E} = -\frac{\psi(-1)\tau(\psi)}{2\pi i\phi(Q)}\left\{\int_{a+iU}^{b_1+iU} + \int_{b_1+iU}^{b_1-iU} + \int_{b_1-iU}^{a-iU}\right\}\frac{L'}{L}(s, \chi\bar{\psi}) \\ \times L(s, \omega'_0)L(s, \omega''_0)\left(\frac{TQ}{2\pi}\right)^s\frac{ds}{s} + O\left(\frac{|\tau(\psi)|}{\phi(Q)}\frac{TQ\log^2(TQ)}{U}\right).$$

Next we evaluate the integrals in $\mathcal{E}$. In view of inequality $\phi(Q) \gg Q^{1-\varepsilon}$ we obtain

$$\frac{|\tau(\psi)|}{\phi(Q)} \ll Q^{-\frac{1}{2}+\varepsilon}. \quad (2.22)$$

By this and by formulae (2.2), (2.13) we have that the first and the third integrals in $\mathcal{E}$ are bounded by

$$\ll Q^{-\frac{1}{2}+\varepsilon}(\log([q, Q]U))^2(QU)^{\frac{3}{8}+\varepsilon}TQ/U.$$

Next we consider the second integral in $\mathcal{E}$. We split it into three parts

$$-\frac{1}{2\pi i}\left\{\int_{b_1-iU}^{b_1-i} + \int_{b_1-i}^{b_1+i} + \int_{b_1+i}^{b_1+iU}\right\}\frac{L'}{L}(s, \chi\bar{\psi})L(s, \omega'_0)L(s, \omega''_0)\left(\frac{TQ}{2\pi}\right)^s\frac{ds}{s}.$$

Again, using formulae (2.2) and (2.13) we see that the first and the third integrals in the last formula are bounded by

$$\ll Q^{-\frac{1}{2}+\varepsilon}(\log([q, Q]U))^2(QU)^{\frac{3}{8}+\varepsilon}(TQ)^{b_1}$$

and the second integral is bounded by

$$\ll Q^{-\frac{1}{2}+\varepsilon}(TQ)^{b_1}(\log([q, Q]))^2Q^{\frac{3}{8}+\varepsilon}.$$

Now we choose $U = T^{1-b_1}$. By $Q \ll \log^A T$ and $q \ll \log^B T$ we obtain that

$$\mathcal{E} \ll T^{1-\frac{c}{\log^{\frac{3}{4}+\varepsilon} T}}.$$

In the same way we get

$$T_1 = \delta(q, Q)\chi(-1)\tau(\chi\psi_0)\frac{1}{\phi(Q)}\frac{TQ}{2\pi}L^2(1, \chi\bar{\psi}) + O\left(T^{1-\frac{c}{\log^{\frac{3}{4}+\varepsilon} T}}\right)$$

and

$$T_2 \ll T^{1 - \frac{c}{\log^{\frac{3}{4}} + \varepsilon} T}.$$

Under GRH we choose $b_1 = 1/2 + 1/\log(QT)$, $U = QT$ and make use of the bound (2.4). This finishes the proof of Lemma 2.2. $\square$

Lemma 2.3. *Let $\psi \bmod Q$ be a primitive Dirichlet character. Then, for $x \rightarrow \infty$,*

$$\begin{aligned} & \sum_{mn \leq x} \psi(m)\psi(n) \exp\left(-2\pi i \frac{mn}{Q}\right) \\ &= -\psi(-1)\tau(\psi) \frac{\phi(Q)}{Q^2} x \left(\log \frac{x}{e} + 2\gamma + 2 \sum_{p|Q} \frac{\log p}{p-1} \right) + O(Q^{-\frac{1}{8}+\varepsilon} x^{\frac{11}{16}+\varepsilon} + Q^4) \end{aligned}$$

uniformly in Q . Under GRH the error term can be replaced by $O(Q^{-1/2+\varepsilon} x^{1/2+\varepsilon} + Q^4)$ uniformly in Q .

Proof. By the orthogonality of Dirichlet characters the sum in Lemma 2.3 can be written as

$$\begin{aligned} S &= \sum_{mn \leq x} \psi(m)\psi(n) \exp\left(-2\pi i \frac{mn}{Q}\right) \\ &= \frac{1}{\phi^2(Q)} \sum_{\eta \bmod Q} \sum_{\omega \bmod Q} \sum_{a=1}^Q \bar{\eta}(a)\psi(a) \sum_{b=1}^Q \bar{\omega}(b)\psi(b) \exp\left(-2\pi i \frac{ab}{Q}\right) \sum_{mn \leq x} \eta(m)\omega(n) \\ &= \psi(-1)\tau(\psi) \frac{1}{\phi(Q)} \sum_{mn \leq x} \eta_0(m)\omega_0(n) + O(\phi^4(Q)). \end{aligned}$$

Perron's formula yields

$$\sum_{mn \leq x} \eta_0(n)\omega_0(l) = -\frac{1}{2\pi i} \int_{a-iU}^{a+iU} L(s, \eta_0)L(s, \omega_0)x^s \frac{ds}{s} + O\left(\frac{x \log^2 x}{U}\right),$$

where $a = 1/\log x$. Let $b_1 = 1/2 + 1/\log x$. Shifting the line of integration we get

$$\begin{aligned} & \sum_{mn \leq x} \eta_0(n)\omega_0(l) \\ &= -\operatorname{Res}_{s=1} L(s, \eta_0)L(s, \omega_0)x^s \frac{1}{s} \\ &+ \frac{1}{2\pi i} \left\{ \int_{a+iU}^{b_1+iU} + \int_{b_1+iU}^{b_1-iU} + \int_{b_1-iU}^{a-iU} \right\} L(s, \eta_0)L(s, \omega_0)x^s \frac{ds}{s} \\ &+ O\left(\frac{x \log^2 x}{U}\right), \end{aligned}$$

The definition of residue, formulae (2.20) and (2.21) lead to

$$\operatorname{Res}_{s=1} L(s, \eta_0) L(s, \omega_0) x^s \frac{1}{s} = \frac{\phi^2(Q)}{Q^2} x \left(\log \frac{x}{e} + 2\gamma + 2 \sum_{p|Q} \frac{\log p}{p-1} \right).$$

Hence, we have

$$S = -\psi(-1)\tau(\psi) \frac{\phi(Q)}{Q^2} x \left(\log(x/e) + 2 \left(\gamma + \sum_{p|Q} \frac{\log p}{p-1} \right) \right) + O(\phi^4(Q)) + \mathcal{E},$$

where

$$\begin{aligned} \mathcal{E} = & \psi(-1)\tau(\psi) \frac{1}{2\pi i \phi(Q)} \left\{ \int_{a+iU}^{b_1+iU} + \int_{b_1+iU}^{b_1-iU} + \int_{b_1-iU}^{a-iU} \right\} L(s, \eta_0) L(s, \omega_0) x^s \frac{ds}{s} \\ & + O\left(\frac{|\tau(\psi)|}{\phi(Q)} \frac{x \log^2 x}{U} \right). \end{aligned}$$

It remains to bound $\mathcal{E}$. By formulae (2.2) and (2.22) we have that the first and the last integrals in $\mathcal{E}$ are bounded by $\ll Q^{-1/2+\varepsilon} (QU)^{3/8+\varepsilon} x/U$. The second integral in $\mathcal{E}$ we split into three parts

$$\left\{ \int_{b_1-iU}^{b_1-i} + \int_{b_1-i}^{b_1+i} + \int_{b_1+i}^{b_1+iU} \right\} L(s, \eta_0) L(s, \omega_0) x^s \frac{ds}{s}.$$

In the last formula the first and the third integrals are both bounded by $\ll Q^{-1/2+\varepsilon} (QU)^{3/8+\varepsilon} x^{1/2}$ and the second integral is bounded by $\ll Q^{-1/2+\varepsilon} x^{1/2}$. We choose $U = x^{1/2}$. Then

$$\mathcal{E} \ll Q^{-\frac{1}{8}+\varepsilon} x^{\frac{11}{16}+\varepsilon}.$$

If we assume GRH then the bound (2.4) gives $\mathcal{E} \ll Q^{-1/2+\varepsilon} x^{1/2+\varepsilon}$. Lemma 2.3 is proved. $\square$

3. Proof of Theorem 1.1

Let $\chi \bmod q$ and $\psi \bmod Q$ be primitive Dirichlet characters such that $\chi \neq \psi$. The proof of the theorem relies on the method of [4]. The idea is to interpret the sum in question as a sum of residues, respectively a contour integral

$$\sum_{0 < \gamma_\chi \leq T} L(\rho_\chi, \psi) L(1 - \rho_\chi, \bar{\psi}) = \frac{1}{2\pi i} \int_{\mathfrak{C}} \frac{L'}{L}(s, \chi) L(s, \psi) L(1-s, \bar{\psi}) ds, \quad (3.1)$$

which can be evaluated by the modified Gonek lemma (Lemma 2.1). We choose an appropriate path of integration $\mathfrak{C}$. In view of the bound for zeros (2.11) we can choose $\frac{1}{\log q} \ll b \leq 1$ and $T \geq 2$ such that

$$\min_{\gamma_\chi} |b - \gamma_\chi| \gg \frac{1}{\log q} \quad \text{and} \quad \min_{\gamma_\chi} |T - \gamma_\chi| \gg \frac{1}{\log(qT)}. \quad (3.2)$$

Let $a = 1 + 1/\log(QT)$ and define the contour $\mathfrak{C}$ to be the rectangle with vertices $a + ib$, $a + iT$, $1 - a + iT$, $1 - a + ib$. Then

$$\begin{aligned} & \frac{1}{2\pi i} \int_{\mathfrak{C}} \frac{L'}{L}(s, \chi) L(s, \psi) L(1 - s, \bar{\psi}) ds \\ &= \frac{1}{2\pi i} \left\{ \int_{a+ib}^{a+iT} + \int_{a+iT}^{1-a+iT} + \int_{1-a+iT}^{1-a+ib} + \int_{1-a+ib}^{a+ib} \right\} \frac{L'}{L}(s, \chi) L(s, \psi) L(1 - s, \bar{\psi}) ds \\ &=: \sum_{j=1}^4 \mathcal{J}_j. \end{aligned} \quad (3.3)$$

First we consider

$$\mathcal{J}_1 = \frac{1}{2\pi i} \int_{a+ib}^{a+iT} \frac{L'}{L}(s, \chi) L(s, \psi) L(1 - s, \bar{\psi}) ds + O(1).$$

By applying the functional equation (2.1) we get

$$\begin{aligned} \mathcal{J}_1 &= \tau(\bar{\psi}) \frac{1}{Q} \frac{1}{2\pi i} \int_{a+ib}^{a+iT} \left(\frac{Q}{2\pi} \right)^s \Gamma(s) \exp\left(-\frac{\pi i s}{2}\right) \frac{L'}{L}(s, \chi) L^2(s, \psi) ds \\ &\quad + \tau(\bar{\psi}) \psi(-1) \frac{1}{Q} \frac{1}{2\pi i} \int_{a+ib}^{a+iT} \left(\frac{Q}{2\pi} \right)^s \Gamma(s) \exp\left(\frac{\pi i s}{2}\right) \frac{L'}{L}(s, \chi) L^2(s, \psi) ds \\ &= \mathcal{J}_{11} + \mathcal{J}_{12}. \end{aligned}$$

In view of the modified Gonek lemma (Lemma 2.1) we have

$$\mathcal{J}_{11} = -\tau(\bar{\psi}) \frac{1}{Q} \sum_{mnl \leq \frac{TQ}{2\pi}} \Lambda(m) \chi(m) \psi(n) \psi(l) \exp\left(-2\pi i \frac{mnl}{Q}\right) + O((QT)^{\frac{1}{2}+\varepsilon})$$

and $\mathcal{J}_{12} = O(Q^{\frac{1}{2}})$. By this, Lemma 2.2, and the equality

$$\tau(\bar{\psi}) \psi(-1) \tau(\psi) = Q, \quad (3.4)$$

we obtain

$$\begin{aligned} \mathcal{J}_1 &= \frac{\phi(Q)}{Q} \frac{L'}{L}(1, \chi \bar{\psi}) \frac{T}{2\pi} \left(\log \frac{TQ}{2\pi e} + 2\gamma + 2 \sum_{p|Q} \frac{\log p}{p-1} + \left(\frac{L''}{L'} - \frac{L'}{L} \right) (1, \chi \bar{\psi}) \right) \\ &\quad - \delta(q, Q) (\chi \psi_0) (-1) \tau(\chi \psi_0) \frac{\tau(\bar{\psi})}{\phi(Q)} \frac{T}{2\pi} L^2(1, \chi \bar{\psi}) + O\left(T^{1-\frac{c}{\log^{\frac{3}{4}+\varepsilon} T}}\right). \end{aligned} \quad (3.5)$$

Under GRH the error term in (3.5) can be replaced by $O(q^{1+\varepsilon} Q^\varepsilon T^{\frac{1}{2}+\varepsilon} + q Q^{9/2+\varepsilon} T^\varepsilon + (QT)^{1/2+\varepsilon})$.

Second we consider $\mathcal{J}_2$ and $\mathcal{J}_4$. Using bounds (2.2), (2.3), and (2.13) we get

$$\begin{aligned} \mathcal{J}_2 &= \frac{1}{2\pi i} \int_{a+iT}^{1-a+iT} \frac{L'}{L}(s, \chi) L(s, \psi) L(1 - s, \bar{\psi}) ds \\ &= O((QT)^{11/16+\varepsilon} \log^2(qT)). \end{aligned} \quad (3.6)$$

Similarly,

$$\mathcal{J}_4 \ll (Q)^{11/16+\varepsilon} \log^2 q. \quad (3.7)$$

Under GRH we use (2.4) instead of (2.2) and we obtain $\mathcal{J}_2, \mathcal{J}_4 \ll (QT)^{1/2+\varepsilon} \log^2(qT)$.

Next we consider $\mathcal{J}_3$. A change of variables $s \mapsto 1 - \bar{s}$ gives

$$\mathcal{J}_3 = -\frac{1}{2\pi i} \int_{a+ib}^{a+iT} \frac{L'}{L}(1 - \bar{s}, \chi) L(1 - \bar{s}, \psi) L(\bar{s}, \bar{\psi}) ds.$$

By complex conjugation we get

$$\overline{\mathcal{J}_3} = -\frac{1}{2\pi i} \int_{a+ib}^{a+iT} \frac{L'}{L}(1 - s, \bar{\chi}) L(1 - s, \bar{\psi}) L(s, \psi) ds.$$

The functional equation (2.1) and its logarithmic derivative (2.9) together with the property (2.8) lead to

$$\begin{aligned} & \frac{L'}{L}(1 - s, \bar{\chi}) L(1 - s, \bar{\psi}) \\ &= \left(\frac{\Delta'}{\Delta}(s, \chi) - \frac{L'}{L}(s, \chi) \right) \\ & \quad \times \tau(\bar{\psi}) \frac{1}{Q} \left(\frac{Q}{2\pi} \right)^s \Gamma(s) \left(\exp\left(-\frac{\pi i s}{2}\right) + \psi(-1) \exp\left(+\frac{\pi i s}{2}\right) \right) L(s, \psi). \end{aligned}$$

Then

$$\begin{aligned} \overline{\mathcal{J}_3} &= -\tau(\bar{\psi}) \frac{1}{2\pi i Q} \int_{a+ib}^{a+iT} \frac{\Delta'}{\Delta}(s, \chi) \left(\frac{Q}{2\pi} \right)^s \Gamma(s) \exp\left(-\frac{\pi i s}{2}\right) L^2(s, \psi) ds \\ & \quad - \psi(-1) \tau(\bar{\psi}) \frac{1}{2\pi i Q} \int_{a+ib}^{a+iT} \frac{\Delta'}{\Delta}(s, \chi) \left(\frac{Q}{2\pi} \right)^s \Gamma(s) \exp\left(+\frac{\pi i s}{2}\right) L^2(s, \psi) ds \\ & \quad + \tau(\bar{\psi}) \frac{1}{2\pi i Q} \int_{a+ib}^{a+iT} \left(\frac{Q}{2\pi} \right)^s \Gamma(s) \exp\left(-\frac{\pi i s}{2}\right) \frac{L'}{L}(s, \chi) L^2(s, \psi) ds \\ & \quad + \psi(-1) \tau(\bar{\psi}) \frac{1}{2\pi i Q} \int_{a+ib}^{a+iT} \left(\frac{Q}{2\pi} \right)^s \Gamma(s) \exp\left(+\frac{\pi i s}{2}\right) \frac{L'}{L}(s, \chi) L^2(s, \psi) ds \\ &= \sum_{j=1}^4 \mathcal{F}_j, \end{aligned} \quad (3.8)$$

say.

First we consider $\mathcal{F}_1$. We rewrite $\mathcal{F}_1$ in the following way:

$$\begin{aligned} \mathcal{F}_1 &= -\frac{1}{Q} \tau(\bar{\psi}) \int_b^T \frac{\Delta'}{\Delta}(a + i\tau, \chi) d \\ & \quad \times \left(\frac{1}{2\pi i} \int_{a+i}^{a+i\tau} \left(\frac{Q}{2\pi} \right)^s \Gamma(s) \exp\left(-\frac{\pi i s}{2}\right) L^2(s, \psi) ds \right). \end{aligned}$$

By Lemmas 2.1 and 2.3 we get

$$\begin{aligned}
 & \frac{1}{2\pi i} \int_{a+i}^{a+i\tau} \left(\frac{Q}{2\pi}\right)^s \Gamma(s) \exp\left(-\frac{\pi i s}{2}\right) L^2(s, \psi) ds \\
 &= \sum_{mn \leq \frac{\tau Q}{2\pi}} \psi(m)\psi(n) \exp\left(-2\pi i \frac{mn}{Q}\right) + O((Q\tau)^{\frac{1}{2}+\varepsilon}) \\
 &= \psi(-1)\tau(\psi) \frac{\phi(Q)}{Q} \frac{\tau}{2\pi} \left(\log \frac{\tau Q}{2\pi e} + 2\gamma + 2 \sum_{p|Q} \frac{\log p}{p-1} \right) + O(Q^{-\frac{1}{8}+\varepsilon} \tau^{\frac{11}{16}+\varepsilon} + Q^4).
 \end{aligned}$$

Hence, in view of the asymptotic formula (2.8) for the logarithmic derivative of the delta function and the property (3.4) of the Gauss sum we obtain

$$\begin{aligned}
 \mathcal{F}_1 &= -\frac{\phi(Q)}{Q} \int_b^T \left(-\log \frac{\tau q}{2\pi} + O\left(\frac{1}{\tau}\right) \right) \\
 &\quad \times d \left(\frac{\tau}{2\pi} \left(\log \frac{\tau Q}{2\pi e} + 2\gamma + 2 \sum_{p|Q} \frac{\log p}{p-1} \right) + O(Q^{-\frac{1}{8}+\varepsilon} \tau^{\frac{11}{16}+\varepsilon} + Q^4) \right) \\
 &= \frac{\phi(Q)}{Q} \frac{T}{2\pi} \left(\log \frac{Tq}{2\pi e} \log \frac{TQ}{2\pi e} + 2 \log \frac{Tq}{2\pi e} \left(\gamma + \sum_{p|Q} \frac{\log p}{p-1} \right) + 1 \right) \\
 &\quad + O(Q^{-\frac{1}{8}+\varepsilon} T^{\frac{11}{16}+\varepsilon} \log q + Q^4 \log(qT)). \tag{3.9}
 \end{aligned}$$

Under GRH the error term for $\mathcal{F}_1$ can be replaced by $O(Q^{-1/2+\varepsilon} T^{1/2+\varepsilon} \log q + Q^4 \log(qT))$.

Reasoning similarly to $\mathcal{F}_1$ we obtain

$$\begin{aligned}
 \mathcal{F}_2 &= -\frac{1}{Q} \psi(-1)\tau(\bar{\psi}) \int_b^T \frac{\Delta'}{\Delta}(a+i\tau, \chi) \\
 &\quad \times d \left(\frac{1}{2\pi i} \int_{a+i}^{a+i\tau} \left(\frac{Q}{2\pi}\right)^s \Gamma(s) \exp\left(\frac{\pi i s}{2}\right) L(s, \psi) ds \right) \\
 &= O(Q^{\frac{1}{2}} \log(qT)). \tag{3.10}
 \end{aligned}$$

Under GRH for $\mathcal{F}_2$ we use the same bound as in (3.10).

We turn to the integral $\mathcal{F}_3$. Using Lemmas 2.1 and 2.2 we get

$$\begin{aligned}
 \mathcal{F}_3 &= -\frac{1}{Q} \tau(\bar{\psi}) \sum_{mnl \leq \frac{\tau Q}{2\pi}} \Lambda(m)\chi(m)\psi(n)\psi(l) \exp\left(-2\pi i \frac{mnl}{Q}\right) + O(Q^{\frac{1}{2}} T^{\frac{1}{2}+\varepsilon}) \\
 &= \frac{\phi(Q)}{Q} \frac{L'}{L}(1, \chi\bar{\psi}) \frac{T}{2\pi} \left(\log \frac{TQ}{2\pi e} + 2\gamma + 2 \sum_{p|Q} \frac{\log p}{p-1} + \left(\frac{L''}{L'} - \frac{L'}{L} \right) (1, \chi\bar{\psi}) \right) \\
 &\quad - \delta(q, Q)(\chi\psi_0)(-1)\tau(\chi\psi_0) \frac{\tau(\bar{\psi})}{\phi(Q)} \frac{T}{2\pi} L^2(1, \chi\bar{\psi}) + O\left(T^{1-\frac{c}{\log 4^{\frac{3}{4}+\varepsilon} T}}\right). \tag{3.11}
 \end{aligned}$$

Under GRH the error term can be replaced by $O(q^{1+\varepsilon} Q^\varepsilon T^{\frac{1}{2}+\varepsilon} + q Q^{9/2+\varepsilon} T^\varepsilon + (QT)^{1/2+\varepsilon})$.

In a similar way as above we have

$$\mathcal{F}_4 \ll Q^{\frac{1}{2}} \log(qT). \quad (3.12)$$

The last bound is used also under GRH.

Now in view of (3.1), (3.3), (3.5)–(3.12) we obtain main asymptotic formula (1.1). Accordingly, the notes after each formula (3.1), (3.3), (3.5)–(3.12) give the error term under GRH. By this Theorem 1.1 is proved.

4. Proof of Corollary 1.4

Let

$$S_1 := \sum_{0 < \gamma_\chi \leq T} L(\rho_\chi, \psi) \quad \text{and} \quad S_2 := \sum_{0 < \gamma_\chi \leq T} |L(\rho_\chi, \psi)|^2.$$

By Hölder's inequality we get

$$\sum_{\substack{0 < \gamma_\chi \leq T \\ L(1/2 + \gamma_\chi, \psi) \neq 0}} 1 \geq \frac{|S_1|^2}{S_2}. \quad (4.1)$$

To evaluate sums S_1 and S_2 we will use Theorems 1.3 and 1.1. First we state several helpful inequalities. From [6, Sec. 14, formulae (11) and (13)], for a non-principal Dirichlet character $\chi \bmod q$, we have

$$|L(1, \chi)| \ll \log q, \quad |L'(1, \chi)| \ll \log^2 q, \quad |L''(1, \chi)| \ll \log^3 q.$$

By Siegel's theorem (see [6, Sec. 21]), for a real primitive Dirichlet character $\chi \bmod q$, we have

$$|L(1, \chi)| \gg q^{-\varepsilon}. \quad (4.2)$$

If $\chi \bmod q$ is a complex primitive character we use the zero free region of $L(s, \chi)$ (see [6, Sec. 14, formula (6)]) and follow the proof of [15, Corollary 7] to obtain the lower bound

$$|L(1, \chi)| \gg q^{-\varepsilon}. \quad (4.3)$$

We will show that the lower bounds (4.2) and (4.3) are valid also for an imprimitive character χ . Indeed, if χ is the imprimitive character induced by a primitive character χ_1 then

$$L(s, \chi) = L(s, \chi_1) \prod_{p|q} \left(1 - \frac{\chi_1(p)}{p^s}\right).$$

Thus for imprimitive character the bounds (4.2) and (4.3) follow by

$$\left| \prod_{p|q} \left(1 - \frac{\chi_1(p)}{p}\right) \right| \geq \prod_{p|q} \left(1 - \frac{1}{p}\right) = \frac{\phi(q)}{q} \gg q^{-\varepsilon}.$$

By above we have that

$$\delta(q, Q)L(1, \chi\bar{\psi})\psi(-1)\tau(\psi)\frac{\tau(\bar{\chi}\psi_0)}{\phi(Q)}\frac{T}{2\pi} \ll TQ^{\frac{1}{2}}\log Q \ll T\log^{1-\varepsilon} T$$

and

$$\frac{L'}{L}(1, \psi\bar{\chi})\frac{T}{2\pi} \ll (qQ)^{\varepsilon}T \ll T\log^{\varepsilon} T.$$

This and Theorem 1.3 give that

$$S_1 \gg T\log T.$$

We will find an upper bound for $|S_2|$. We have

$$\sum_{p|Q} \frac{\log p}{p-1} \leq \log Q \sum_{p \leq Q} \frac{1}{p-1} \ll \log^{\varepsilon} T,$$

$$\begin{aligned} & \delta(q, Q)(\tau(\chi\psi_0)\tau(\bar{\psi})L^2(1, \chi\bar{\psi}) + \tau(\bar{\chi}\psi_0)\tau(\psi)L^2(1, \bar{\chi}\psi))\frac{(\chi\psi_0)(-1)}{\phi(Q)}\frac{T}{2\pi} \\ & \ll Q^{\frac{1}{2}}T\log^2 Q \ll T\log T, \end{aligned}$$

and

$$\frac{L''}{L}(1, \psi\bar{\chi})\frac{T}{2\pi} \ll (qQ)^{\varepsilon}T \ll T\log T.$$

Then Theorem 1.1 leads to

$$S_2 \ll \frac{\phi(Q)}{Q}T\log^2 T.$$

Now the corollary follows by formula (4.1).

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