Note on zeros of the derivative of the Selberg zeta-function

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Abstract. Wenzhi Luo studied the distribution of nontrivial zeros of the derivatives of Selberg zeta-functions on cocompact hyperbolic surfaces, and obtained an asymptotic formula for zero density with bounded height. Then he related the distribution of zeros to the multiplicities of Laplacian eigenvalues. Using better bounds for the growth of Selberg zeta-functions we improve some of the above results.

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1. Introduction

Let F denote a compact Riemann surface of genus $g \geq 2$. F can be represented as a quotient space $\Gamma \backslash H$, where $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ is a strictly hyperbolic Fuchsian group and H is the upper half-plane. The Γ conjugacy class determined by $P \in \Gamma$ will be denoted by $\{P\}$ and its norm by $N\{P\}$. By P_0 will be denoted the primitive element of Γ . The Selberg zeta-function for $\sigma > 1$ is given by (Hejhal [2])

$$Z(s) = \prod_{\{P_0\}} \prod_{k=0}^{\infty} (1 - N(P_0)^{-s-k}). \quad (1)$$

It is an entire function with a functional equation

$$Z(s) = f(s)Z(1-s), \quad (2)$$

where

$$f(s) = \exp \left(4\pi(g-1) \int_0^{s-\frac{1}{2}} v \tan(\pi v) dv \right).$$

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All nontrivial (nonreal) zeros of $Z(s)$ lie on the critical line $\sigma = 1/2$. From the functional equation by the Phragmén-Lindelöf principle it follows the bound (see Proposition 7.2 in [2])

$$Z(\sigma + it) \ll \exp\left(\frac{3}{2}\text{area}(\Gamma \setminus H)t\right) \quad \text{for} \quad \sigma \geq -1, t \geq 0. \quad (3)$$

In [4] W. Luo studied the distribution of nontrivial zeros of the derivative of the Selberg zeta function $Z'(s)$. He calculated an asymptotic formula for the number of zeros up to the height T . The bound (3) allowed to reach the accuracy $O(T)$ in this asymptotic formula. Luo showed that $Z'(s)$ has only finitely many nontrivial zeros (expected no zeros) to the left from the critical line. Then he related the distribution of the zeros to the multiplicities of Laplacian eigenvalues. Note, that absence of zeros to the left of the critical line of the derivative of the Riemann zeta function is equivalent to RH (Speiser [5], Levinson and Montgomery [3]).

It is well known that the Riemann hypothesis for the Riemann zeta-function yields the Lindelöf hypothesis (Titchmarsh [7], §13.5). In [1] we showed that analogous relation holds for the Selberg zeta-function. Since the Riemann hypothesis for $Z(s)$ is valid, in [1] we derived the following Lindelöf type bound: for any $\varepsilon > 0$

$$Z\left(\frac{1}{2} + it\right) \ll \exp(\varepsilon t). \quad (4)$$

By using this bound it is possible to improve error terms in Luo's asymptotic formulas. This is the aim of our paper.

In the next section we state theorems and third section contains proofs.

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2. Results

In this paper T always tends to plus infinity.

Define the number of nontrivial zeros of $Z^{(r)}(s)$ by

$$N_r(T) = \#\{\beta + i\gamma : Z^{(r)}(\beta + i\gamma) = 0, 0 < \gamma \leq T\}.$$

Let $N(P_0) = \min_{P_0} \{N(P_0)\}$.

Theorem 1.

$$N_1(T) = \frac{\text{area}(\Gamma \setminus H)}{2} T^2 - T \log N(P_0) + o(T).$$

For the number of nontrivial zeros of $Z(s)$ a better error term is known to be (Hejhal [2])

$$N_0(T) = \frac{\text{area}(\Gamma \setminus H)}{2} T^2 + O\left(\frac{T}{\log T}\right). \quad (5)$$

The next theorem shows that many nontrivial zeros $\varrho = \beta + i\gamma$ of the derivative of the Selberg zeta-function lie off the critical line, but not very far. Let

$$m_0 = \#\{P : N(P) = N(P_{00})\} \quad \text{and} \quad \Lambda(P_{00}) = \frac{\log N(P_{00})}{1 - N(P_{00})^{-1}}.$$

Theorem 2.

$$\sum_{1 < \gamma \leq T} \left(\beta - \frac{1}{2}\right) = T \log T + T \log \frac{m_0 \Lambda(P_{00})}{(\text{area}(\Gamma \setminus H) - 1) \sqrt{N(P_{00})}} + o(T).$$

Define the number of nontrivial zeros of $Z'(s)$ to the right-hand side from σ by

$$N_1(\sigma, T) = \#\{\beta + i\gamma : Z'(\beta + i\gamma) = 0, \beta > \sigma, 0 < \gamma \leq T\}.$$

From Theorem 2 it follows

Corollary 3. *For any $\varepsilon > 0$*

$$N_1\left(\frac{1}{2} + \varepsilon, T + 1\right) - N_1\left(\frac{1}{2} + \varepsilon, T\right) = o(T).$$

The corollary can also be derived from Proposition 2 of [1], using the bound $Z'(1/2 + it) \ll \exp(\varepsilon T)$ (see Lemma 7 below). Moreover, Proposition 2 of [1] shows that the last bound is equivalent to Corollary 3. This is an analogy to Backlund's equivalent for the Lindelöf hypothesis for the Riemann zeta-function (Titchmarsh [7], §13.5).

Nontrivial zeros $s_j = 1/2 \pm it_j$ of $Z(s) = Z_\Gamma(s)$ correspond to, with the same multiplicities, eigenvalues

$$0 < \lambda_j = s_j(1 - s_j) = 1/4 + t_j^2$$

of the hyperbolic Laplacian Δ on $X = \Gamma \setminus H$ (Heijhal [2]). Luo [4] proved that on the critical line $\Re s = 1/2$, $Z'(s)$ *only* vanishes at (multiple) zeros of $Z(s)$ except possibly at $s = 1/2$. This, (5), and Theorem 1 give a relation for the number of distinct eigenvalues of the Laplacian Δ up to the height T :

Theorem 4.

$$\#\{t_j : 0 < t_j \leq T\} = N_1\left(\frac{1}{2}, T\right) + T \log N(P_{00}) + o(T).$$

Here ordinates t_j are counted without multiplicities.

From the above we see that almost all nontrivial zeros of $Z'(s)$ lie in a very narrow region to the right-hand side of the critical line:

Corollary 5. *For any fixed $\varepsilon > 0$,*

$$\#\{\beta + i\gamma : Z'(\beta + i\gamma) = 0, 0 < \beta < \frac{1}{2} + \frac{\log^{1+\varepsilon} T}{T}, 0 < \gamma \leq T\} \sim N_1(T).$$

Theorems 1, 2 and 4 with error terms $O(T)$ were obtained by Luo [4]. The next section is devoted to the proofs of Theorems 1 and 2.

3. Proofs

We need to know the behavior of $f(s)$ for large t , where $f(s)$ is the factor from the functional equation (2).

Lemma 6. For $t > 0$,

$$f(s) = \exp\left(2\pi i(g-1)\left(s - \frac{1}{2}\right)^2 + \frac{\pi i(g-1)}{12} + O\left(\frac{\sigma^2 + |\sigma t|}{e^{2\pi t}}\right)\right)$$

uniformly in σ .

Proof. In view of the definition (2) of $f(s)$ we calculate the following integral

$$\begin{aligned} \int_0^{s-\frac{1}{2}} v \tan(\pi v) dv &= i \int_0^{s-\frac{1}{2}} v \frac{1 - e^{2\pi i v}}{1 + e^{2\pi i v}} dv \\ &= i \int_0^{s-\frac{1}{2}} v dv - i \left(\int_0^{it} + \int_{it}^{\sigma-1/2+it} \right) \frac{v}{1 + e^{-2\pi i v}} dv \\ &= \frac{i}{2} \left(s - \frac{1}{2}\right)^2 + i \int_0^t \frac{x}{1 + e^{2\pi x}} dx - \int_0^{\sigma-1/2} \frac{(x+it)}{1 + e^{2\pi t - 2\pi i x}} dx \\ &= \frac{i}{2} \left(s - \frac{1}{2}\right)^2 + \frac{i}{48} + O\left(\frac{\sigma^2 + |\sigma t|}{e^{2\pi t}}\right). \end{aligned}$$

Here we used

$$\int_0^\infty \frac{x}{1 + e^{2\pi x}} dx = \frac{1}{48}.$$

Next we obtain the Lindelöf type bound for $Z'(s)$.

Lemma 7. Let $\varepsilon > 0$ and $\sigma_0 > 1$. Then

$$Z'(\sigma + iT) \ll \begin{cases} \exp(\varepsilon T) & \text{for } \frac{1}{2} \leq \sigma \leq \sigma_0, \\ \exp\left(\left(\frac{1}{2} - \sigma + \varepsilon\right)T\right) & \text{for } \sigma \leq \frac{1}{2}. \end{cases}$$

Proof. In [1] we obtained that for any $\varepsilon > 0$, $Z(\sigma + iT) \ll \exp(\varepsilon T)$ if $1/2 \leq \sigma \leq \sigma_0$. From the functional equation (2) and Lemma 6 we find

$$Z(\sigma + iT) \ll \exp\left(\left(\frac{1}{2} - \sigma + \varepsilon\right)T\right) \quad \text{for } \sigma \leq \frac{1}{2}.$$

Cauchy formula gives

$$Z'(s) = \frac{1}{2\pi i} \int_C \frac{Z(z)}{(z-s)^2} dz,$$

where C is any circle with the center at s . If we take the circle of radius $= \delta < \varepsilon$, then for $1/2 \leq \sigma \leq \sigma_0$

$$Z'(\sigma + iT) \ll \frac{e^{(\delta+\varepsilon)T}}{\delta} \ll e^{2\varepsilon T}.$$

Now we turn to $\sigma \leq 1/2$. Differentiating the functional equation (2) we get

$$Z'(s) = f(s)(Z(1-s)\text{area}(\Gamma \backslash H)(\frac{1}{2}-s)\cot(\pi s) - Z'(1-s)). \quad (6)$$

In view of

$$\cot(\pi(\sigma + iT)) = -i(1 + O(e^{-2\pi T}))$$

and Lemma 6 the remaining part of the lemma follows.

Following proposition is the main tool in the investigation of zeros.

Proposition 8. *Let $0 < a < 1/2$ and $2 < t_0 < 3$. Then*

$$\sum_{\substack{\beta > a \\ t_0 < \gamma \leq T}} (\beta - a) = \left(\frac{1}{2} - a\right) \frac{\text{area}(\Gamma \backslash H)}{2} T^2 + T \log T - T(\log(m_0 \Lambda(P_{00})) - a \log N(P_{00})) + T(\log(\text{area}(\Gamma \backslash H)) - 1) + o(T).$$

Proof. Put

$$X(s) = \frac{Z'(s)}{m_0 \Lambda(P_{00}) N(P_{00})^{-s}}.$$

Obviously, zeros of $X(s)$ correspond exactly to zeros of $Z'(s)$. Luo [4] showed that $Z'(s)$ has finitely many nontrivial zeros in the halfplane $\sigma < 1/2$ and has no zeros in $\sigma > \sigma_0$ for some sufficiently large $\sigma_0 > 2$. Let $T \geq 4$. We choose $0 < a < 1/2$, $2 < t_0 < 3$ and $|T' - T| = o(1)$ such that $Z'(s)$ does not vanish on the boundary of the rectangle R with vertices $a + t_0 i$, $\sigma_0 + t_0 i$, $\sigma_0 + T' i$, $a + T' i$. Applying Littlewood's lemma (Titchmarsh [7], §9.9) we get

$$\begin{aligned} 2\pi \sum_{\substack{\beta > a \\ t_0 < \gamma \leq T'}} (\beta - a) &= \int_{t_0}^{T'} \log |X(a + it)| dt - \int_{t_0}^{T'} \log |X(\sigma_0 + it)| dt - \\ &\quad - \int_a^{\sigma_0} \arg X(\sigma + it_0) d\sigma + \int_a^{\sigma_0} \arg X(\sigma + iT) d\sigma = \\ &= \sum_{j=1}^4 I_j. \end{aligned} \quad (7)$$

Here $\arg X(s)$ is defined by continuous variation starting at $s = \sigma_0$ with $\arg X(\sigma_0) = 0$, then along the lines connecting σ_0 with $\sigma_0 + it$ and $\sigma_0 + it$ with $\sigma + it$, provided that the path does not cross a zero of $X(s)$; if it does, we put

$$\arg X(s) = \lim_{\varepsilon \rightarrow +0} X(\sigma + it + \varepsilon).$$

Luo [4] obtained

$$I_2, I_3 = O(1),$$

and

$$I_1 = \left(\frac{1}{2} - a\right) \frac{\text{area}(\Gamma \setminus H)}{2} T^2 - T(\log(m_0 \Lambda(P_{00})) - a \log N(P_{00})) \\ + \int_{t_0}^{T'} \log |Z(1 - a - it)| dt + O(1). \quad (8)$$

Now we will prove that $I_4 = o(T)$ and later we will return to I_1 .

We split the integral

$$I_4 = \int_a^{\sigma_0} \arg X(\sigma + iT') d\sigma = \int_a^{\frac{1}{2}} + \int_{\frac{1}{2}}^{\sigma_0} = I_{41} + I_{42}.$$

First we will evaluate I_{42} . For this we will show, that for $\sigma \geq 1/2$

$$\arg X(\sigma + iT') = o(T). \quad (9)$$

Let σ_0 be large enough, such that $\Re X(s) \neq 0$ for $\sigma > \sigma_0$. Suppose that $\Re X(\sigma + iT')$ has N zeros for $1/2 \leq \sigma \leq \sigma_0$. Then divide $[1/2, \sigma_0]$ into at most $N+1$ subintervals in each of which $\Re X(\sigma + iT')$ is of constant sign. Then

$$|\arg X(\sigma + iT')| \leq (N+1)\pi.$$

To estimate N let

$$g(z) := \frac{1}{2} (X(z + iT') + X(z - iT')).$$

Then we have $g(\sigma) = \Re X(\sigma + iT)$. Let $n(r)$ denote the number of zeros of $g(z)$ in $|z - \sigma_0| \leq r$. Then

$$|\arg X(\sigma + iT')| \leq \left(n\left(c - \frac{1}{2}\right) + 1 \right) \pi.$$

By Jensen's Theorem (Titchmarsh [6], §3.61.)

$$\int_0^R \frac{n(r)}{r} dr = \frac{1}{2\pi} \int_0^{2\pi} \log |g(\sigma_0 + Re^{i\theta})| d\theta - \log |g(\sigma_0)|. \quad (10)$$

Let $R = \sigma_0 - 1/2 + \delta$, $\delta > 0$. By Lemma 7, for large T ,

$$\log |g(\sigma_0 + Re^{i\theta})| < \begin{cases} \varepsilon T & \text{for } \Re(\sigma_0 + Re^{i\theta}) \geq \frac{1}{2}, \\ (\delta + \varepsilon)T & \text{for } \frac{1}{2} - \delta \leq \Re(\sigma_0 + Re^{i\theta}) \leq \frac{1}{2}. \end{cases}$$

The length of the arc of the circle is $|s - \sigma_0| = R$, which is to the left of the critical line $\sigma = 1/2$, is $O(\delta^{1/2})$. Now we see that the right-hand side of (10) is at most

$$O(\varepsilon T) + O(\delta^{1/2}(\delta + \varepsilon)T).$$

Since

$$\frac{\delta}{R} n(R - \delta) \leq \int_0^R \frac{n(r)}{r} dr$$

we conclude that

$$n(R - \delta) = O\left(\frac{\varepsilon}{\delta} + \delta^{-\frac{1}{2}}(\delta + \varepsilon)T\right)$$

and taking $\delta = \varepsilon^{2/3}$ we obtain that $n(R - \delta) = O(\varepsilon^{1/3}T)$. The bound (9) is proved. Thus

$$I_{42} = o(T).$$

Now we evaluate the integral I_{41} . By the functional equation (see (6)) we have

$$X(s) = \frac{f(s)N(P_{00})^s}{m_0\Lambda(P_{00})} (Z(1-s)\text{area}(\Gamma\backslash H)(\frac{1}{2}-s)\cot(\pi s) - Z'(1-s)).$$

Thus

$$I_{41} = \int_{-b}^{\frac{1}{2}} \arg \frac{f(\sigma + iT)N(P_{00})^{\sigma+iT}}{m_0\Lambda(P_{00})} \left(Z(1-s)\text{area}(\Gamma\backslash H)(\frac{1}{2}-s)\cot(\pi s) - Z'(1-s) \right) d\sigma.$$

From above we have, that

$$\begin{aligned} & \arg \frac{f(\frac{1}{2} + iT)N(P_{00})^{\frac{1}{2}+iT}}{m_0\Lambda(P_{00})} (Z(\frac{1}{2} - iT)\text{area}(\Gamma\backslash H)(-iT)\cot(\pi(\frac{1}{2} + iT)) - Z'(\frac{1}{2} - iT)) \\ &= \arg X(\frac{1}{2} + iT) = o(T). \end{aligned}$$

Lemma 6 yields

$$\arg \frac{f(s)N(P_{00})^s}{m_0\Lambda(P_{00})} \Big|_{\frac{1}{2}+iT}^{\sigma+iT} = O(1),$$

and by Lemma 7 we see that for $a \leq \sigma \leq 1/2$

$$Z(1-s)\text{area}(\Gamma\backslash H)(\frac{1}{2}-s)\cot(\pi s) - Z'(1-s) \ll \exp(\varepsilon t).$$

By applying reasoning similar to that used in case of I_{42} we obtain that $I_{41} = o(T)$ and thus $I_4 = o(T)$.

We return to the integral I_1 . In view of (8) it remains to evaluate the integral

$$I_{11} := \int_{t_0}^{T'} \log |Z(1-a-it)| dt.$$

When $\Re(s) \geq 2$ we have (see Proposition 4.13(c) in Hejhal [2]) $X(s) = 1 + O(c_T^\sigma)$, for some c_T with $0 < c_T < 1$. Applying Cauchy's theorem to $\log Z(s)$ we obtain

$$\begin{aligned} I_{11} &:= \int_{1-a}^{\infty} \arg Z(\sigma - it_0) d\sigma - \int_{1-a}^{\infty} \arg Z(\sigma - iT') d\sigma \\ &= - \int_{1-a}^2 \arg Z(\sigma - iT') d\sigma + O(1). \end{aligned}$$

Repeating the steps used in I_{42} we get $I_{11} = o(T)$. By this Proposition 8 follows.

Proof of Theorem 1. Remembering that $Z'(s)$ can have only finitely many nontrivial zeros in $\sigma < 1/2$, we write

$$N_1(T) = \frac{2}{a} \left(\sum_{0 < \gamma < T} \left(\beta - \frac{a}{2} \right) - \sum_{0 < \gamma < T} (\beta - a) \right).$$

Now using Proposition 8 we obtain the theorem.

Proof of Theorem 2 follows from Theorem 1 and Proposition 8 in view of

$$\sum_{1 < \gamma < T} \left(\beta - \frac{1}{2} \right) = \sum_{1 < \gamma < T} (\beta - a) - \left(\frac{1}{2} - a \right) \sum_{1 < \gamma < T} 1.$$

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