THE a -VALUES OF THE SELBERG ZETA-FUNCTION

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Abstract. We study the roots (a -values) of $Z(s) = a$, where $Z(s)$ is the Selberg zeta-function attached to a compact Riemann surface. We obtain an asymptotic formula for the number of nontrivial a -values. If $a \neq 0$, we show that the analogue of the Riemann hypothesis fails for nontrivial a -values; on other hand, almost all nontrivial a -values are arbitrarily close to the critical line. We also compare distributions of a -values for the Selberg and the Riemann zeta-functions.

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1 MOTIVATION

Let $s = \sigma + it$ be a complex variable. We start with the Riemann zeta-function, which is given, for $\sigma > 1$, by the following Dirichlet series or Euler product:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

The Lindelöf hypothesis (LH) states that, for any $\varepsilon > 0$,

$$\zeta\left(\frac{1}{2} + it\right) \ll_{\varepsilon} |t|^{\varepsilon} \quad (|t| \rightarrow \infty).$$

In [2], we extended the Backlund equivalent for the LH to the following statement:

Let a be a complex number. The Lindelöf hypothesis for $\zeta(s)$ is true if and only if, for any $\sigma' > 1/2$, the number of roots of $\zeta(s) - a$ in the region $\sigma > \sigma'$, $T \leq t \leq T + 1$ is $o(\log T)$ as $T \rightarrow \infty$.

The original Backlund equivalent (see [1] or [8, Sect. 13.5]) corresponds to the case $a = 0$. Here and further, the numbers of roots are always counted according multiplicities. Solutions of $f(s) = a$ are called a -values of $f(s)$.

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We see that the Lindelöf hypothesis follows from the Riemann hypothesis, which states that the Riemann zeta-function has no zeros to the right of the critical line $\sigma = 1/2$.

Moreover, in [2], we proved that the above extended Backlund equivalent can be generalized to some other functions, for example, to the Selberg zeta-function $Z(s)$ attached to a compact Riemann surface (for the full definition, see the next section). In this case, we have the following equivalent:

Let a be a complex number. For any $\varepsilon > 0$,

$$Z\left(\frac{1}{2} + it\right) \ll_{\varepsilon} \exp(\varepsilon|t|) \quad (|t| \rightarrow \infty) \quad (1.1)$$

if and only if, for any $\sigma' > 1/2$, the number of roots of $Z(s) - a$ in the region $\sigma > \sigma'$, $T \leq t \leq T + 1$ is $o(T)$ as $T \rightarrow \infty$.

The bound (1.1) corresponds to the LH for the Riemann zeta-function. For the Selberg zeta-function, the analog of RH is true (see [3, Chap. 2, Thm. 4.11]); thus, in the last equivalence, both statements are true.

The Selberg and the Riemann zeta-functions have many similar properties: the functional equation, the Euler product, similar Selberg trace and Weil explicit formulas. There are many results about a -values of $\zeta(s)$ (see [4, 6, 8]). In view of the above, it would be of interest to compare the a -value distribution for both zeta-functions. In the remaining part of this paper, we consider the a -values of the Selberg zeta-function.

2 STATEMENT OF RESULTS

Let F denote a compact Riemann surface of genus $g \geq 2$. The surface F can be represented as a quotient space $\Gamma \backslash H$, where $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ is a strictly hyperbolic Fuchsian group, and H is the upper half-plane. The Γ conjugacy class determined by $P \in \Gamma$ will be denoted by $\{P\}$, and its norm is defined by $N(\{P\}) = N(P) = \alpha^2$ if the eigenvalues of P are α and α^{-1} ($|\alpha| > 1$). By P_0 we denote the primitive element of Γ . The Selberg zeta-function for $\sigma > 1$ is given by Hejhal [3, Chap. 2, Def. 4.1]:

$$Z(s) = \prod_{\{P_0\}} \prod_{k=0}^{\infty} (1 - N(P_0)^{-s-k}). \quad (2.1)$$

By the Selberg trace formula the Selberg zeta-function is extended to an entire function (see [3, Chap. 2, Thm. 4.11]), which satisfies the functional equation (see [3, Chap. 2, Thm. 4.12])

$$Z(s) = X(s)Z(1-s), \quad (2.2)$$

where

$$X(s) = \exp\left(4\pi(g-1) \int_0^{s-1/2} v \tan(\pi v) dv\right). \quad (2.3)$$

The next lemma shows the behavior of $X(s)$ for large t (see [3, Chap. 2, formula (4.4)]).

Lemma 1. *For $t > 1$,*

$$\begin{aligned} X(s) = \exp\left(2\pi i(g-1)\left(s - \frac{1}{2}\right)^2 + \frac{\pi i(g-1)}{6} + O\left(\frac{t}{e^{2\pi t}}\right) + O\left(\frac{(\sigma - 1/2)^2}{e^{2\pi t}}\right) \right. \\ \left. + O\left(\frac{(\sigma - 1/2)t}{e^{2\pi t}}\right)\right) \quad (t \rightarrow \infty) \end{aligned}$$

uniformly in σ .

The lemma will be proved in the next section.

In this paper, a is always a fixed complex number, and T always tends to $+\infty$. By β_a and γ_a we denote the real and imaginary parts of the a -value. Let

$$N(P_{00}) = \min_{P_0} \{N(P_0)\} \quad \text{and} \quad m_0 = \#\{\{P_0\}: N(P_0) = N(P_{00})\}.$$

By the definition of the Selberg zeta-function (2.1) we find that

$$Z(s) = 1 + m_0 N(P_{00})^{-\sigma} + o(N(P_{00})^{-\sigma}) \quad (\sigma \rightarrow \infty). \quad (2.4)$$

Note that $N(P_{00}) > 1$ (see [3, Chap. 2, Sect. 2]). Thus, there exists a number $A > 1$, depending on a , such that $Z(s) \neq a$ for $\sigma \geq A$. Further, by $\overline{Z(s)} = Z(\bar{s})$, the functional equation (2.2), and by Lemma 1 we see that there is $\tau \geq 0$ such that

$$Z(s) \neq a \quad \text{for } \sigma \leq 1 - A \text{ and } |t| \geq \tau. \quad (2.5)$$

In view of this, we say that an a -value $\varrho_a = \beta_a + i\gamma_a$ is *nontrivial* if $1 - A < \beta_a < A$. If $\beta_a \leq 1 - A$ (and $|\gamma_a| < \tau$), then $\varrho_a = \beta_a + i\gamma_a$ is called a *trivial a -value*.

The following proposition will be our main tool in the investigation of a -values.

Proposition 1. *Let τ be defined by formula (2.5). Let $b = b(T) = o(\log T)$ be a positive and unboundedly increasing function. Then*

$$\sum_{\tau < \gamma_a \leq T} (\beta_a + b) = (g - 1) \left(\frac{1}{2} + b \right) T^2 - \frac{T}{2\pi} \log |1 - a| + o(T)$$

if $a \neq 1$, and

$$\sum_{\tau < \gamma_1 \leq T} (\beta_1 + b) = (g - 1) \left(\frac{1}{2} + b \right) T^2 - \frac{bT}{2\pi} \log N(P_{00}) - \frac{T}{2\pi} \log m_0 + o(T)$$

otherwise.

The proposition will be proved in the next section. The accuracy $o(T)$ is due to the bound (1.1).

Define by $N(a, T)$ the number of nontrivial a -values with $\tau < \gamma_a \leq T$. In Proposition 1, subtracting the case b from $b + 1$, we obtain the following theorem.

Theorem 1. *For $a \neq 1$,*

$$N(a, T) = (g - 1)T^2 + o(T),$$

and, for $a = 1$,

$$N(1, T) = (g - 1)T^2 - \frac{T}{2\pi} \log N(P_{00}) + o(T).$$

If $a = 0$, then the last theorem is true with a better error term $O(T/\log T)$ (see [3, Sect. 2.8, Thm. 8.19]).

The expression

$$\sum_{\tau < \gamma_a \leq T} \left(\beta_a - \frac{1}{2} \right) = \sum_{\tau < \gamma_a \leq T} (\beta_a + b) - \left(\frac{1}{2} + b \right) \sum_{\tau < \gamma_a \leq T} 1,$$

Theorem 1, and Proposition 1 with appropriate b lead to the following statement.

Theorem 2. Let τ be defined by formula (2.5). For $a \neq 1$,

$$\sum_{\tau < \gamma_a \leq T} \left(\beta_a - \frac{1}{2} \right) = -\frac{T}{2\pi} \log |1 - a| + o(T),$$

and, for $a = 1$,

$$\sum_{\tau < \gamma_1 \leq T} \left(\beta_1 - \frac{1}{2} \right) = \frac{T}{4\pi} \log \frac{N(P_{00})}{m_0^2} + o(T).$$

If $a \neq 0$, then we see that many a -values lie off the critical line $\sigma = 1/2$ and are distributed asymmetrically with respect to this line.

Let $N^+(a, \delta, T)$ and $N^-(a, \delta, T)$ be the numbers of nontrivial a -values in the corresponding regions $\sigma > 1/2 + \delta$, $1 < t \leq T$ and $\sigma < 1/2 - \delta$, $1 < t \leq T$. Let

$$N_0(a, \delta, T) := N(a, T) - (N^-(a, \delta, T) + N^+(a, \delta, T)).$$

The next theorem shows that almost all a -values are arbitrarily close to the critical line.

Theorem 3. For

$$\delta = \frac{(\log \log T)^2}{\log T},$$

we have

$$N^-(a, \delta, T) + N^+(a, \delta, T) \ll \frac{T^2}{\log \log T}$$

and

$$N_0(a, \delta, T) = (g - 1)T^2 + O\left(\frac{T^2}{\log \log T}\right).$$

The theorem will be proved in the next section.

The obtained results show a similar distribution of a -values for the Riemann and the Selberg zeta-functions. If we replace $(g - 1)T^2$ by $1/(2\pi)T \log T$ and $o(T)$ by $O(\log T)$, we obtain statements nearly identical to the ones proved by Levinson [4] for the Riemann zeta-function. The difference is that, in his proof, Levinson used the mean value bound for $\zeta(s)$, while here we use the Lindelöf-type bound (1.1).

The next section contains the proofs of Lemma 1, Proposition 1, and Theorem 3.

3 PROOFS

Proof of Lemma 1. In view of definition (2.3) of $X(s)$, we calculate the following integral:

$$\begin{aligned} \int_0^{s-1/2} v \tan(\pi v) dv &= i \int_0^{s-1/2} v \frac{1 - e^{2\pi i v}}{1 + e^{2\pi i v}} dv = i \int_0^{s-1/2} v dv - i \left(\int_0^{it} + \int_{it}^{\sigma-1/2+it} \right) \frac{2v}{1 + e^{-2\pi i v}} dv \\ &= \frac{i}{2} \left(s - \frac{1}{2} \right)^2 + 2i \int_0^t \frac{x}{1 + e^{2\pi x}} dx - 2i \int_0^{\sigma-1/2} \frac{(x + it)}{1 + e^{2\pi t - 2\pi i x}} dx \\ &= \frac{i}{2} \left(s - \frac{1}{2} \right)^2 + \frac{i}{24} + O\left(\frac{t}{e^{2\pi t}}\right) + O\left(\frac{(\sigma - 1/2)^2}{e^{2\pi t}}\right) + O\left(\frac{(\sigma - 1/2)t}{e^{2\pi t}}\right) \quad (t \rightarrow \infty). \end{aligned}$$

Here we used the fact that

$$\int_0^{\infty} \frac{x}{1 + e^{2\pi x}} dx = \frac{1}{48}.$$

This proves the lemma. $\square$

Proof of Proposition 1. First, we consider the case $a \neq 1$. Put

$$G(s) = \frac{Z(s) - a}{1 - a}.$$

Obviously, the zeros of $G(s)$ correspond exactly to the a -values of $Z(s)$. Let

$$c = \log T. \quad (3.1)$$

Recall that $b = b(T)$ is a positive and unboundedly increasing function. Thus, for sufficiently large T , we have $Z(s) \neq a$ if $\sigma \leq -b$ or $\sigma \geq c$. We assume that $Z(s) \neq a$ on the boundaries of the rectangle R with vertices $c + i\tau'$, $c + iT'$, $-b + iT'$, $-b + i\tau'$, where $T < T' < T + 1/T$ and $\tau < \tau' < \tau + 1$. Applying Littlewood's lemma (see [8, Sect. 9.9]) to the function $G(s)$ on the rectangle R , we get

$$\begin{aligned} 2\pi \sum_{\substack{\beta_a > -b \\ \tau' < \gamma_a \leq T'}} (\beta_a + b) &= \int_{\tau'}^{T'} \log |G(-b + it)| dt - \int_{\tau'}^{T'} \log |G(c + it)| dt \\ &\quad - \int_{-b}^c \arg G(\sigma + i\tau') d\sigma + \int_{-b}^c \arg G(\sigma + iT') d\sigma =: \sum_{j=1}^4 I_j. \end{aligned} \quad (3.2)$$

Here $\arg G(s)$ is defined by continuous variation starting at $s = A + 1$, then along lines connecting $A + 1$ with $A + 1 + it$ and $A + 1 + it$ with $\sigma + it$, provided that the path does not cross a zero of $G(s)$; if it does, we put

$$\arg G(s) = \lim_{\varepsilon \rightarrow +0} \arg G(\sigma + it + \varepsilon).$$

In view of (2.4), we choose the branch of argument such that $\arg G(s)$ tends to zero as $\sigma \rightarrow \infty$. Then $\arg Z(A + 1) = 0$.

By the functional equation (2.2), Lemma 1, and formula (2.4) we get

$$\begin{aligned} I_1 &= \int_{\tau'}^{T'} \log \left| \frac{X(b + it)Z(1 + b - it) - a}{1 - a} \right| dt \\ &= 4\pi(g - 1) \int_{\tau'}^{T'} \left(t \left(\frac{1}{2} + b \right) + O\left(\frac{\sigma^2 + |\sigma t|}{e^{2\pi t}} \right) \right) dt \\ &\quad + \int_{\tau'}^{T'} \log \left| Z(1 + b - it) - \frac{a}{X(-b + it)} \right| dt - \int_{\tau'}^{T'} \log |1 - a| dt \\ &= 2\pi(g - 1)T'^2 \left(\frac{1}{2} + b \right) - T' \log |1 - a| + O\left(1 + \frac{T'}{(\min\{N(P_{00}), e\})^b} \right). \end{aligned} \quad (3.3)$$

Similarly, and even simpler, we get

$$I_2 = O\left(\frac{T'}{N(P_{00})^c}\right) = o(T). \quad (3.4)$$

Now we consider the integral I_4 . We split I_4 into two integrals

$$I_4 = \int_{-b}^c \arg G(\sigma + iT') d\sigma = \int_{-b}^{1/2} + \int_{1/2}^c = I_{41} + I_{42}. \quad (3.5)$$

First, we will evaluate I_{42} . In view of formula (2.4) and definition (3.1), we can choose a large fixed positive number c' such that

$$I_{42} = \int_{1/2}^c \arg G(\sigma + iT') d\sigma = \int_{1/2}^{c'} \arg G(\sigma + iT') d\sigma + O(\log T').$$

We will show that, for $\sigma \geq 1/2$,

$$\arg G(\sigma + iT') = o(T'). \quad (3.6)$$

Let c' be large enough, such that $\Re G(s) \neq 0$ for $\sigma > c'$. Suppose that $\Re G(\sigma + iT')$ has N zeros for $1/2 \leq \sigma \leq c'$. Then divide $[1/2, c']$ into at most $N + 1$ subintervals in each of which $\Re G(\sigma + iT')$ is of constant sign. Then

$$|\arg G(\sigma + iT')| \leq (N + 1)\pi.$$

To estimate N , let

$$g(z) := \frac{1}{2}(G(z + iT') + \overline{G(\bar{z} + iT')}).$$

Then we have $g(\sigma) = \Re G(\sigma + iT')$. Let $n(r)$ denote the number of zeros of $g(z)$ in $|z - c'| \leq r$. Then

$$|\arg G(\sigma + iT')| \leq \left(n\left(c' - \frac{1}{2}\right) + 1\right)\pi.$$

By Jensen's theorem (see [7, Sect. 3.61]),

$$\int_0^R \frac{n(r)}{r} dr = \frac{1}{2\pi} \int_0^{2\pi} \log |g(c' + Re^{i\theta})| d\theta - \log |g(c')|. \quad (3.7)$$

Let $R = c' - 1/2 + \delta$, $\delta > 0$. We have that, for any $\varepsilon > 0$ and $\sigma \geq 1/2$ (see [2], comments below Theorem 5),

$$Z(\sigma + iT') \ll \exp(\varepsilon T'). \quad (3.8)$$

Thus, for $\Re(c' + Re^{i\theta}) \geq 1/2$ and large T' ,

$$\log |g(c' + Re^{i\theta})| < 2\varepsilon T'.$$

Then the functional equation (2.2) and Lemma 1 give that there is an absolute constant $d > 0$ such that

$$\log |g(c' + Re^{i\theta})| < d(\delta + \varepsilon)T'.$$

The length of the arc of the circle $|s - c'| = R$, which is to the left of the critical line $\sigma = 1/2$, is

$$2R \arcsin \frac{\sqrt{2R\delta - \delta^2}}{R} = O(\sqrt{R\delta}) = O(\sqrt{\delta}) \quad (\delta \rightarrow 0).$$

Now we see that the right-hand side of (3.7) is at most

$$O(\varepsilon T') + O(\delta^{1/2}(\delta + \varepsilon)T').$$

Since

$$\frac{\delta}{R}n(R - \delta) \leq \int_0^R \frac{n(r)}{r} dr,$$

we conclude that

$$n(R - \delta) = O\left(\frac{\varepsilon}{\delta}T' + \delta^{-1/2}(\delta + \varepsilon)T'\right),$$

and, taking $\delta = \varepsilon^{2/3}$, we obtain that $n(R - \delta) = O(\varepsilon^{1/3}T')$. The bound (3.6) is proved. Hence,

$$I_{42} = o(T').$$

We return to the integral I_{41} defined by Eq. (3.5). To evaluate I_{41} , we will use the functional equation (2.2). First, we choose the branch of $\arg X(\sigma + iT')$. The functional equation (2.2) and the equality $\arg Z(1/2 - iT') = -\arg Z(1/2 + iT')$ give that

$$\arg X\left(\frac{1}{2} + iT'\right) \equiv 2 \arg Z\left(\frac{1}{2} + iT'\right) \pmod{2\pi}.$$

We choose

$$\arg X\left(\frac{1}{2} + iT'\right) = 2 \arg Z\left(\frac{1}{2} + iT'\right), \quad (3.9)$$

and we define $\arg X(\sigma + iT')$ by continuous variation along the segment connecting $1/2 + iT'$ with $\sigma + iT'$. Applying functional equation (2.2), we get

$$\begin{aligned} I_{41} &= \int_{-b}^{1/2} \arg X(\sigma + iT') d\sigma + \int_{-b}^{1/2} \arg \left(Z(1 - \sigma - iT') - \frac{a}{X(\sigma + iT')} \right) d\sigma \\ &\quad - \int_{-b}^{1/2} \arg(1 - a) d\sigma = I_{411} + I_{412} + o(\log T'). \end{aligned}$$

It is known (see [3, Chap. 2, completion of the proof of Thm. 8.1] or [5, Thm. 1]) that $\arg Z(1/2 + iT') = O(T'/\log T')$. Then Eq. (3.9) gives that

$$\arg X\left(\frac{1}{2} + iT'\right) = O\left(\frac{T'}{\log T'}\right). \quad (3.10)$$

Further, by Lemma 1 we get

$$\arg X(\sigma + iT') = 2\pi(g-1)\left(\sigma - \frac{1}{2}\right)^2 - T'^2 + 2\pi k + O\left(\frac{\sigma^2 + |\sigma T'|}{e^{2\pi T'}}\right), \quad (3.11)$$

where the constant $2\pi k$ with integer k depends on the branch of $\arg X(\sigma + iT')$. By (3.10) and (3.11) we have that $2\pi k = T'^2 + O(T'/\log T')$. This and (3.11) give

$$I_{411} = O(b^3) + O\left(\frac{bT'}{\log T'}\right) = o(T).$$

Arguing similarly as for I_{42} , we obtain that $I_{412} = o(T)$. Thus, $I_4 = o(T)$, and similarly, $I_3 = o(\log T)$. From this and from (3.2)–(3.4) Proposition 1 follows for $a \neq 1$. For $a = 1$, we prove the proposition, in a similar way, considering the function

$$G^*(s) = \frac{Z(s) - 1}{m_0/N(P_{00})^s}.$$

This gives Proposition 1. $\square$

To prove Theorem 3, we need the following lemma.

Lemma 2. *We have*

$$\sum_{\substack{1 < \gamma_a \leq T \\ \beta_a > \frac{1}{2}}} \left(\beta_a - \frac{1}{2}\right) \ll \frac{T^2 \log \log T}{\log T}.$$

Proof. By Theorem 1,

$$\begin{aligned} \sum_{\substack{1 < \gamma_a \leq T \\ \beta_a > \frac{1}{2}}} \left(\beta_a - \frac{1}{2}\right) &= \sum_{\substack{1 < \gamma_a \leq T \\ \beta_a > \frac{1}{2} + \frac{\log \log T}{\log T}}} \left(\beta_a - \frac{1}{2} - \frac{\log \log T}{\log T}\right) \\ &\quad + \sum_{\substack{1 < \gamma_a \leq T \\ \beta_a > \frac{1}{2} + \frac{\log \log T}{\log T}}} \frac{\log \log T}{\log T} + \sum_{\substack{1 < \gamma_a \leq T \\ \frac{1}{2} < \beta_a \leq \frac{1}{2} + \frac{\log \log T}{\log T}}} \left(\beta_a - \frac{1}{2}\right) \\ &= \sum_{\substack{1 < \gamma_a \leq T \\ \beta_a > \frac{1}{2} + \frac{\log \log T}{\log T}}} \left(\beta_a - \frac{1}{2} - \frac{\log \log T}{\log T}\right) + O\left(\frac{T^2 \log \log T}{\log T}\right). \end{aligned}$$

Arguing along the same lines as in the proof of Proposition 1, only with $b = -1/2 - (\log \log T)/\log T$, we

obtain

$$\begin{aligned} \sum_{\substack{1 < \gamma_a \leq T \\ \beta_a > \frac{1}{2} + \frac{\log \log T}{\log T}}} \left(\beta_a - \frac{1}{2} - \frac{\log \log T}{\log T} \right) &= \int_1^T \log \left| Z \left(\frac{1}{2} + \frac{\log \log T}{\log T} + it \right) - a \right| dt + O(T) \\ &= \int_{\sqrt{T}}^T \log \left| Z \left(\frac{1}{2} + \frac{\log \log T}{\log T} + it \right) - a \right| dt + O(T), \end{aligned}$$

where, in the last step, we used the bound (3.8). By Hejhal [3, Chap. 2, Prop. 10.10 and Thm. 8.1] it follows that

$$\log |Z(\sigma + it)| \ll \left(\frac{t}{\log t} \right)^{2 \max(0, 1-\sigma)} \log t \quad \text{for } \sigma \geq \frac{1}{2} + \frac{1}{\log(t/\log t)}.$$

Hence, for sufficiently large T_0 , there is a positive constant C such that, for $T > T_0$ and $\sqrt{T} \leq t \leq T$, we have

$$\log \left| Z \left(\frac{1}{2} + \frac{\log \log T}{\log T} + it \right) - a \right| < C \frac{T}{(\log T)^2}.$$

This proves Lemma 2. $\square$

Proof of Theorem 3. By Lemma 2 we have that

$$N^+(a, \delta, T) \leq \frac{\log T}{(\log \log T)^2} \sum_{\substack{1 < \gamma_a \leq T \\ \beta_a > \frac{1}{2} + \frac{\log \log T}{\log T}}} \left(\beta_a - \frac{1}{2} \right) \ll \frac{T^2}{\log \log T}.$$

Next, we consider a bound for

$$N^-(a, \delta, T) = \sum_{\substack{1 < \gamma_a \leq T \\ \beta_a < \frac{1}{2} - \delta}} 1.$$

Let $b = b(T)$ satisfy the conditions of Proposition 1. If T is sufficiently large, then, for any a -value $\varrho_a = \beta_a + i\gamma_a$, we have that $\beta_a < b$, and thus,

$$\sum_{1 < \gamma_a \leq T} (\beta_a + b) \leq \sum_{\substack{1 < \gamma_a \leq T \\ \beta_a > \frac{1}{2}}} \left(\beta_a - \frac{1}{2} \right) + \left(b + \frac{1}{2} \right) \sum_{\substack{1 < \gamma_a \leq T \\ \beta_a \geq \frac{1}{2} - \delta}} 1 + \left(b + \frac{1}{2} - \delta \right) \sum_{\substack{1 < \gamma_a \leq T \\ \beta_a < \frac{1}{2} - \delta}} 1.$$

The last inequality, together with Lemma 2, Proposition 1, and Theorem 1, yields

$$0 \leq O \left(\frac{T^2 \log \log T}{\log T} \right) - \delta N^-(a, \delta, T).$$

Thus,

$$N^-(a, \delta, T) \ll \frac{T^2}{\log \log T}.$$

The above bounds for $N^+(a, \delta, T)$ and $N^-(a, \delta, T)$ and Theorem 1 give the required formula for $N_0(a, \delta, T)$, which completes the proof. $\square$

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