GROWTH OF THE LERCH ZETA-FUNCTION

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Abstract. It is believed that the Lindelöf hypothesis is also true for the Lerch zeta-function. Here we present results supporting this conjecture. We first consider the growth of the Lerch zeta-function assuming the generalized Lindelöf hypothesis for Dirichlet L -functions. We next prove that Huxley's exponent $32/205$ in the Lindelöf hypothesis for the Riemann zeta-function holds also for the Lerch zeta-function.

Keywords: Lindelöf hypothesis, growth of the critical line, Lerch zeta-function, Dirichlet L -function.

1. INTRODUCTION

Let $s = \sigma + it$ be a complex variable. Denote by $\{\lambda\}$ the fractional part of a real number λ . We write $f(x) = O(g(x))$ or $f(x) \ll g(x)$ if

$$\limsup_{x \rightarrow \infty} \frac{|f(x)|}{g(x)}$$

is finite and $f(x) = o(g(x))$ if this limit equals 0. Further, the notation $f(x) \asymp g(x)$ denotes that the relations $g(x) \ll |f(x)| \ll g(x)$ hold. For $a \not\equiv 0 \pmod{q}$, the notation $\bar{a}$ means that $a\bar{a} \equiv 1 \pmod{q}$.

For $\sigma := \Re s > 1$, the Lerch zeta-function is given by the series

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m + \alpha)^s}$$

with parameters $\lambda \in \mathbb{R}$ and $0 < \alpha \leq 1$. This function can be analytically continued to the whole complex plane with a possible exception of a pole at the point $s = 1$. Since $L(\lambda, \alpha, s)$ is periodic in λ , we assume that $0 < \lambda \leq 1$.

In [2], the relationship between the Lindelöf hypothesis and the zero distribution for Lerch zeta-functions is discussed. The Lindelöf hypothesis for the Lerch zeta-function states that

$$L\left(\lambda, \alpha, \frac{1}{2} + it\right) \ll |t|^\varepsilon$$

as $t \rightarrow \pm\infty$ for every $\varepsilon > 0$. Dependently on the expected zero distribution, the family of Lerch zeta-functions can be divided into two classes.

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The first class is characterized by the Riemann hypothesis and consists of $L(\lambda, \alpha, s)$ with $\lambda, \alpha \in \{1/2, 1\}$. In this class, we have

$$L(1, 1, s) = \zeta(s), \quad L\left(\frac{1}{2}, 1, s\right) = (1 - 2^{1-s})\zeta(s), \quad L\left(1, \frac{1}{2}, s\right) = (2^s - 1)\zeta(s),$$

where $\zeta(s)$ is the Riemann zeta-function, and

$$L\left(\frac{1}{2}, \frac{1}{2}, s\right) = 2^s L(\chi, s),$$

where $L(\chi, s)$ is the Dirichlet L -function with Dirichlet character $\chi \bmod 4$, $\chi(3) = -1$. All these functions have an Euler product. We denote by $N(\sigma_1, \sigma_2, T)$ the number of zeros in the rectangle $1/2 < \sigma_1 < \sigma < \sigma_2 < 1$, $|t| \leq T$. It is known that, for zeta functions from this class, we have $N(\sigma_1, \sigma_2, T) = o(T)$ for all $1/2 < \sigma_1 < \sigma_2 < 1$, and from the Lindelöf hypothesis it follows that

$$N(\sigma_1, \sigma_2, T) = O(T^{2(1-\sigma_1)}) \quad (\text{Density hypothesis}).$$

The remaining Lerch zeta-functions form the other class. For these functions, one expects that, for every pair of parameters and for all $1/2 < \sigma_1 < \sigma_2 < 1$, we have $N(\sigma_1, \sigma_2, T) \asymp T$. This is proved for all λ if α is a transcendental number and for $\lambda = 1$ if $\alpha \neq 1/2, 1$ is a rational number (see [1]).

In [2], we conjectured that, despite the differences in the zero distribution, the Lindelöf hypothesis should hold for any Lerch zeta-function. In this paper, we present results supporting this conjecture.

The generalized Lindelöf hypothesis (GLH) states that, for all $\varepsilon > 0$, the Dirichlet L -function to the character $\chi \bmod q$ satisfies the bound

$$L\left(\chi, \frac{1}{2} + it\right) \ll |tq|^\varepsilon$$

as $t \rightarrow \pm\infty$. Since, for rational parameters, the Lerch zeta-function can be expressed by Dirichlet L -functions, we obtain:

THEOREM 1. *Assume GLH. Then, for all $\varepsilon > 0$,*

$$L\left(\frac{l}{m}, \frac{k}{q}, \frac{1}{2} + it\right) \ll ((m, q)mq)^{\frac{1}{2}} |tmq|^\varepsilon$$

as $t \rightarrow \pm\infty$.

The Hurwitz zeta-function $\zeta(s, \alpha)$ is a partial case of the Lerch zeta-function, $\zeta(s, \alpha) = L(1, \alpha, s)$. In view of the theorem above and the property that rational numbers very well approximate Liouville numbers, we have:

THEOREM 2. *Assume GLH. Let α be a Liouville number. Then, for all $\varepsilon > 0$ and $\delta > 1$, there exists a sequence $t_n \rightarrow \infty$ such that*

$$\zeta\left(\frac{1}{2} + it, \alpha\right) \ll |t|^\varepsilon$$

for $t \in (t_n^{1/\delta}, t_n^\delta)$.

From the proof and formula (10) (see below) one can see that Theorem 2 can be also proved for $L(\alpha, \alpha, s)$.

In [2], it is shown that, for fixed parameters λ and α ,

$$L\left(\lambda, \alpha, \frac{1}{2} \pm it\right) \ll t^{\frac{1}{6}} \log t.$$

For the Riemann zeta-function, Huxley [5] obtained the bound

$$\zeta\left(\frac{1}{2} + it\right) \ll t^{32/205+\varepsilon}.$$

Applying Huxley's [4], [5] bounds for exponential sums, we prove the following:

THEOREM 3. *For all $\varepsilon \geq 0$,*

$$L(\lambda, \alpha, 1/2 + it) - \frac{1}{\alpha^{1/2+it}} - \left(\frac{2\pi}{t}\right)^{it} e^{it + \frac{\pi i}{4} - 2\pi i \lambda \alpha} \left(\frac{1}{\lambda^{1/2-it}} - \frac{e^{-\pi t + \pi i/2 + 2\pi i \alpha}}{(1 - \{\lambda\})^{1/2-it}} \right) \ll t^{32/205+\varepsilon}$$

as $t \rightarrow \infty$, uniformly in $\lambda, \alpha \in (0, 1]$.

By Theorem 3 and the formula

$$L(s, \chi) = q^{-s} \sum_{n=1}^q \chi(n) \zeta\left(s, \frac{n}{q}\right)$$

we immediately obtain the following:

COROLLARY 4. *For all $\varepsilon \geq 0$,*

$$L\left(\frac{1}{2} + it, \chi\right) \ll \sqrt{q}|t|^{32/205+\varepsilon}$$

as $t \rightarrow \pm\infty$.

The next section is devoted to some preliminary lemmas. Section 3 contains the proofs of the theorems.

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2. LEMMAS

We denote $e(x) = e^{2\pi i x}$. We will need Weyl's differencing lemma (Graham and Kolesnik [3], Lemma A of van der Corput's method):

LEMMA 5. *Let $f(x)$ be a real function, and let A , B , and D be positive integers. Then*

$$\left| \sum_{n=A+1}^B e(f(n)) \right|^2 \leq \left(\frac{B-A}{D} + 1 \right) \left(B-A + \frac{2}{D} \Re \sum_{e=1}^{D-1} \sum_{d=1}^{D-e} \sum_{n=A+1}^{B-d-e} e(f(n+d) - f(n)) \right).$$

Let

$$S_i = \sum_{m=M_1(i)}^{M_2(i)} e\left(T F\left(\frac{m}{M}, y_i\right)\right), \quad (1)$$

where $M \leq M_1(i) \leq M_2(i) < 2M$, and $y_1, \dots, y_I \in [0, 1]$ are such that

$$y_{i+1} - y_i \geq \frac{1}{J} \quad (2)$$

for $i \leq I-1$. We write ∂_1, ∂_2 for $\partial/\partial x, \partial/\partial y$, and F_1 for $\partial_1 F$, etc.

LEMMA 6 (Huxley [4], Theorem 2). Suppose that, for some $\delta > 0$, for $1 - \delta \leq x \leq 2 + \delta$, $0 \leq y \leq 1$, the function $F(x, y)$ is six times differentiable and, for some constants $C_1 \geq 1$ and $C_2 > 0$,

$$|\partial_1^r \partial_2^s F(x, y)| \leq C_1, \quad 2 \leq r \leq 6, \quad 0 \leq s \leq 2, \quad r + s \leq 6,$$

$$1/C_1 \leq |\partial_1^r F(x, y)|, \quad r = 2, 3, \quad (3)$$

and

$$\left| \partial_1 \frac{\partial_1^2 \partial_2 F(x, y)}{\partial_1^3 F(x, y)} \right| \geq C_2. \quad (4)$$

Suppose also that either Case 1 or Case 2 holds:

Case 1: $M \ll T^{1/2}$ and (3) and (4) hold for $r = 4$;

Case 2: $M \gg T^{1/2}$ and

$$|3F_{111}^2 - F_{11}F_{1111}| \geq C_3, \quad |\Delta| \geq C_4$$

for some positive constants C_3 and C_4 , where

$$\Delta = \begin{vmatrix} 3F_{111}^2 + 4F_{11}F_{1111} & 3F_{11}F_{111} & F_{11}^2 \\ F_{11111} & F_{1111} & F_{111} \\ F_{11112} & F_{1112} & F_{112} \end{vmatrix}.$$

Let S_i be the sum defined by (1), and let $y_1, \dots, y_I$ be real numbers satisfying (2). Let $\varepsilon > 0$ be arbitrary. Then, in both cases, for

$$\frac{T^{8/57}}{T^{11/19}} + \frac{J}{I} \gg \frac{T}{M^2} + \frac{M^2}{T}, \quad (5)$$

we have

$$\sum_{i=1}^I |S_i|^5 \ll I^{16/19} M^{5/2} T^{89/114+\varepsilon} + I^{8/11} J^{3/11} M^{5/2} T^{49/66+\varepsilon}. \quad (6)$$

Moreover, there exists a constant C_5 such that, in Case 1, for $M \leq T^{1/2}/C_5$, we have

$$\sum_{i=1}^I |S_i|^5 \ll IM^{5/2} T^{5/6+\varepsilon} E_1^{3/8} \left(1 + \frac{JM^2}{IT} \right)^{1/2} \quad (7)$$

with

$$E_1 = \frac{1}{T^{8/33}} + \frac{T^{2/3}}{M^2} + \left(J + \frac{IT}{M^2} \right)^{-1}$$

and, in Case 2, for $M \geq C_5 T^{1/2}$, we have

$$\sum_{i=1}^I |S_i|^5 \ll IM^{5/2} T^{5/6+\varepsilon} E_2^{3/8} \left(1 + \frac{JT}{IM^2} \right)^{1/2} \quad (8)$$

with

$$E_2 = \frac{1}{T^{8/33}} + \frac{M^2}{T^{4/3}} + \left(J + \frac{IM^2}{T} \right)^{-1}.$$

The constant C_5 and implied constants depend on δ , ε , C_1 , C_2 , C_3 , and C_4 and on the implied constants in the definitions of Cases 1 and 2.

3. PROOFS OF THEOREMS

Proof of Theorem 1. Let ϕ be Euler's totient function, and let

$$D := \{d: d = (jq + k, mq) \text{ for } j \in \{0, 1, \dots, m-1\}\}.$$

By the well-known formula

$$\frac{1}{\phi(q)} \sum_{\chi} \bar{\chi}(a) \chi(n) = \begin{cases} 1 & \text{if } n \equiv a \pmod{q} \text{ and } (a, q) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

we have, for $\sigma > 1$,

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{e^{2\pi i n \frac{L}{m}}}{\left(n + \frac{k}{q}\right)^s} &= \sum_{j=0}^{m-1} e^{2\pi i j \frac{L}{m}} \sum_{n=0}^{\infty} \frac{1}{\left(nm + j + \frac{k}{q}\right)^s} \\ &= q^s \sum_{d \in D} \frac{1}{d^s \phi\left(\frac{mq}{d}\right)} \sum_{\chi \bmod \frac{mq}{d}} \sum_{\substack{j=0 \\ (jq+k, mq)=d}}^{m-1} \bar{\chi}\left(\frac{jq+k}{d}\right) e^{2\pi i j \frac{L}{m}} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \\ &= q^s \sum_{d \in D} \frac{1}{d^s \phi\left(\frac{mq}{d}\right)} \sum_{\chi \bmod \frac{mq}{d}} \tau_d(\chi) L(s, \chi). \end{aligned}$$

Here

$$\tau_d(\chi) =: \sum_{\substack{j=0 \\ (jq+k, mq)=d}}^{m-1} \bar{\chi}\left(\frac{jq+k}{d}\right) e^{2\pi i j \frac{L}{m}}.$$

Let

$$q_1 = \frac{q}{(m, q)}.$$

Denote by $\chi(n; pq)$ the Dirichlet character $\chi(n) \bmod pq$. If $(p, q) = 1$, then $\chi(n; pq) = \chi(n; p)\chi(n; q)$. In view of this notation,

$$\begin{aligned} \tau_1(\chi \bmod qm) &= \sum_{j=0}^{m-1} \bar{\chi}(jq + k; qm) e^{2\pi i j \frac{L}{m}} \\ &= \bar{\chi}(k; q_1) \sum_{j=0}^{m-1} \bar{\chi}(jq + k; (m, q)m) e^{2\pi i j \frac{L}{m}}. \end{aligned}$$

We will show that the above sum, in absolute value, does not exceed $\sqrt{(m, q)m}$. We consider

$$|\tau_1(\chi)|^2 = \sum_{j_1=0}^{m-1} \sum_{j_2=0}^{m-1} \bar{\chi}(j_1q + k; (m, q)m) \chi(j_2q + k; (m, q)m) e^{2\pi i(j_1 - j_2)\frac{l}{m}}.$$

To change the summation variable we treat the congruence

$$j_2q + k \equiv n(j_1q + k) \pmod{(m, q)m}. \quad (9)$$

Define

$$A := \{j \mid 0 \leq j \leq m-1, (jq + k, (m, q)m) = 1\}.$$

For fixed $j_1 \in A$, we denote by N_{j_1} the set of solutions of (9) in $n \in \{0, 1, \dots, (m, q)m - 1\}$ when j_2 runs through the set A . For all $j_1 \in A$, the sets N_{j_1} coincide with each other. To prove this, first note that, for any fixed pair $j_1, j_2 \in A$, the congruence (9) has a unique solution in n and, thus, all sets N_{j_1} have the same number of elements. Next, for different j_1 and j'_1 from A and for every $n_{j_1, j_2} \in N_{j_1}$, we can find $n_{j'_1, j'_2} \in N_{j'_1}$ such that $n_{j_1, j_2} = n_{j'_1, j'_2}$. Indeed, since

$$n_{j_1, j_2} \equiv (j_2q + k)(\overline{j_1q + k}) \pmod{(m, q)m} \quad \text{and} \quad n_{j'_1, j'_2} \equiv (j'_2q + k)(\overline{j'_1q + k}) \pmod{(m, q)m},$$

we can find a suitable $j'_2 \in A$ by the congruence

$$(j_2q + k)(\overline{j'_1q + k}) \equiv (j'_2q + k)(\overline{j_1q + k}) \pmod{(m, q)m}.$$

We denote $N = N_{j_1}$ for any $j_1 \in A$. Note that the set N contains no more than (m, q) elements of the form $n = 1 + rm$, $r \in \mathbb{Z}$.

In view of the above, changing the summation variable and then the summation order, we obtain

$$|\tau_1(\chi)|^2 = \sum_{n \in N} \chi(n; (q, m)m) e^{2\pi i \frac{q\overline{n}(n-1)l}{(m, q)m}} \sum_{j_1=0}^{m-1} e^{2\pi i \frac{j_1(n-1)l}{m}} \leq (m, q)m.$$

It remains to evaluate τ_d for $d \geq 2$. From $(k, q) = 1$ and $d \in D$ we have that $(d, q) = 1$, $d \mid (m/(m, q))$ and $(m/d, q) = (m, q)$. Let j_0 be such that $(j_0q + k, mq) = d$. Then, for $d \in D$, we obtain

$$\tau_d\left(\chi \bmod \frac{m}{d}q\right) = \sum_{\substack{j=0 \\ (jq+k, mq)=d}}^{m-1} \bar{\chi}\left(\frac{jq+k}{d}\right) e^{2\pi i j \frac{l}{m}} = \sum_{r=0}^{\frac{m}{d}-1} \bar{\chi}\left(rq + \frac{j_0q+k}{d}\right) e^{2\pi i r \frac{l}{\frac{m}{d}}}.$$

By the same reasoning as above, we find

$$|\tau_d(\chi)|^2 \leq (m, q) \frac{m}{d}.$$

Now, in view of $\sum_{d \mid m} 1 \ll m^\varepsilon$, we have

$$L\left(\frac{l}{m}, \frac{k}{q}, \frac{1}{2} + it\right) \ll ((m, q)mq)^{\frac{1}{2}} |tmq|^\varepsilon \sum_{d \mid m} \frac{1}{d} \ll ((m, q)mq)^{\frac{1}{2}} |tmq|^\varepsilon.$$

This completes the proof of Theorem 1.

Proof of Theorem 2. For $0 < \sigma_0 \leq \sigma \leq 2$ and $\pi x \geq |t| \geq 2\pi$, we have (Karatsuba and Voronin [7])

$$\zeta(s; \alpha) = \sum_{0 \leq n \leq x} \frac{1}{(n + \alpha)^s} + \frac{x^{1-s}}{s-1} + O_{\sigma_0}(x^{-\sigma}).$$

Thus, if

$$\left| \alpha - \frac{k}{l} \right| \leq \frac{1}{l^a},$$

then, applying the classical formula

$$f(a) - f(b) = f'(\xi)(a - b), \quad a \leq \xi \leq b$$

for real and complex parts, we obtain

$$\left| \zeta\left(\frac{1}{2} + it; \alpha\right) - \zeta\left(\frac{1}{2} + it; \frac{k}{l}\right) \right| \ll \sum_{1 \leq n \leq x} \frac{t}{n^{\frac{3}{2}} l^a} + O(x^{-\frac{1}{2}}) \ll \frac{t}{l^a}.$$

Let us choose $a = (\delta - \varepsilon)/\varepsilon$ and $t_n = l_n^{\delta/\varepsilon}$. Then by Theorem 1 and the estimate

$$\zeta\left(\frac{1}{2} + it; \alpha\right) \ll \zeta\left(\frac{1}{2} + it; \frac{k}{l}\right) + \left| \zeta\left(\frac{1}{2} + it; \alpha\right) - \zeta\left(\frac{1}{2} + it; \frac{k}{l}\right) \right|$$

we obtain Theorem 2.

Proof of Theorem 3. Let $0 < \lambda \leq 1$, $0 < \alpha \leq q1$, and $0 < \sigma \leq q1$. Moreover, let $t \geq q1$, $y = (t/(2\pi))^{1/2}$, $q = [y]$, $k = [y - \alpha]$, and $\beta = q - k$. Then (see [1], Chap. 4, formula (1.11))

$$\begin{aligned} L(\lambda, \alpha, s) &= \sum_{m=0}^k \frac{e(\lambda m)}{(m + \alpha)^s} \\ &+ \left(\frac{2\pi}{t}\right)^{\sigma - \frac{1}{2} + it} e^{it + \frac{\pi i}{4} - 2\pi i \lambda \alpha} \left(\sum_{m=0}^q \frac{e(-\alpha m)}{(m + \lambda)^{1-s}} - \frac{e^{-\pi t + \pi i \sigma + 2\pi i \alpha}}{(1 - \{\lambda\})^{1-s}} \right) \\ &+ \left(\frac{2\pi}{t}\right)^{\frac{\sigma}{2}} e^{if(\lambda, \alpha, \sigma, t)} \psi(2y - 2q - k - \{\lambda\} - \alpha) + O(t^{\frac{\sigma-2}{2}}), \end{aligned} \quad (10)$$

where

$$\begin{aligned} f(\lambda, \alpha, t) &= -\frac{t}{2\pi} \log \frac{t}{2\pi e} - \frac{7}{8} + \frac{1}{2}(\alpha^2 - \{\lambda\}^2) \\ &- \alpha\beta + 2y(\beta + \{\lambda\} - \alpha) - \frac{1}{2}(q + k) - \{\lambda\}(\beta + \alpha) \end{aligned}$$

and

$$\psi(a) = \frac{\cos(\pi(a^2/2 - a - 1/8))}{\cos(\pi a)}.$$

Note that formula (10) holds uniformly in λ and α .

Let

$$f(n) = \lambda n - \frac{T \log(n + \alpha)}{2\pi}.$$

We consider the exponential sum

$$S(M) = \sum_{n=M}^{M_2} e(f(n)), \quad (11)$$

where $M_2 < 2M < T$. Let $M = T^\beta$, $0 < \beta \leq 1$. To prove the theorem, in view of (10), it suffices to show that, for $0 \leq \beta \leq 1/2$ and $\varepsilon > 0$,

$$S(M) \ll T^{\frac{32}{205} + \frac{\beta}{2} + \varepsilon}$$

uniformly in λ , α , and β . The last bound trivially holds for $0 \leq \beta \leq 64/205 = 0.31\dots$. For $0.3 \leq \beta \leq 0.407$, we rewrite $S(M)$ as a double exponential sum and use Huxley's bound (Lemma 6). In the remaining range, we apply the recent bound due to Huxley [5].

By Weyl's differencing lemma (Lemma 5) we have

$$\left| \sum_{n=M}^{M_2} e(f(n)) \right|^2 \leq \left(\frac{M}{D} + 1 \right) \left(M + \frac{2}{D} \Re \sum_{e=1}^{D-1} \sum_{d=1}^{D-e} \sum_{n=M}^{M_2-d-e} e(f(n+d) - f(n)) \right). \quad (12)$$

To evaluate the triple sum in the last formula, we use Lemma 6 with T replaced by TI/M , where I appears below:

$$\begin{aligned} & \sum_{d=1}^{D-e} \sum_{n=M}^{M_2-d-e} e(f(n+d) - f(n)) \\ &= \left(\sum_{\substack{I=2^j, \\ 0 \leq j \leq \frac{\log(D-e)}{\log 2} - 1}} \sum_{I \leq k < 2I} \sum_{n=M}^{M_2-k-e} + \sum_{k=2^{\lfloor \frac{\log(D-e)}{\log 2} \rfloor}}^{D-e} \sum_{n=M}^{M_2-k-e} \right) e(f(n+k) - f(n)) \\ &\ll \sum_{\substack{I=2^j, \\ 0 \leq j \leq \frac{\log(D-e)}{\log 2}}} \sum_{i=1}^I \left| \sum_{n=M}^{M_2-(y_i+1)I-e} e(f(n + (y_i+1)I) - f(n)) \right| \\ &= \sum_{\substack{I=2^j, \\ 0 \leq j \leq \frac{\log(D-e)}{\log 2}}} \sum_{i=1}^I \left| \sum_{n=M}^{M_2-(y_i+1)I-e} e\left(\frac{TI}{M} F\left(\frac{n}{M}, y_i\right)\right) \right| = \sum_{\substack{I=2^j, \\ 0 \leq j \leq \frac{\log(D-e)}{\log 2}}} \sum_{i=1}^I |S_i| \\ &\ll \sum_{\substack{I=2^j, \\ 0 \leq j \leq \frac{\log(D-e)}{\log 2}}} I^{\frac{4}{5}} \left(\sum_{i=1}^I |S_i|^5 \right)^{\frac{1}{5}}; \end{aligned} \quad (13)$$

here we used the notation $y_i = k/I - 1$ and

$$F(x, y) = \frac{M}{TI} (f(Mx + (1+y)I) - f(Mx)).$$

Clearly,

$$0 \leq y_i < 1 \quad \text{and} \quad y_{i+1} - y_i \geq \frac{1}{I}.$$

We check the conditions of Lemma 6. We further suppose that $D = o(M)$. For $r \geq 2$, we have

$$\begin{aligned} \partial_1^r F &= \frac{(-1)^{r-1} M^r (r-2)!}{2\pi I} \left(\frac{1}{(Mx + \alpha)^{r-1}} - \frac{1}{(Mx + (1+y)I + \alpha)^{r-1}} \right) \\ &= \frac{(-1)^{r-1} (r-1)!}{2\pi} \frac{1+y}{(x + (1+y)\theta I/M + \alpha/M)^r} \end{aligned}$$

with some $0 \leq \theta \leq 1$ and, for $r, s \geq 1$,

$$\partial_1^r \partial_2^s F = \frac{(-1)^{r+s} (r-1)! s!}{2\pi} \frac{M^r I^{s-1}}{(Mx + (1+y)I + \alpha)^{r+s-1}}.$$

Thus, in Lemma 6, for sufficiently large T , we can choose $C_1 = 40$. We denote

$$U = \frac{1}{Mx + \alpha}$$

and

$$V = \frac{1}{Mx + (1+y)I + \alpha}.$$

In view of

$$\left| \partial_1 \frac{\partial_1^{r-1} \partial_2 F(x, y)}{\partial_1^r F(x, y)} \right| = \left| \partial_1 \frac{I}{M \left(1 - \left(1 + \frac{(1+y)I}{Mx + \alpha} \right)^{r-1} \right)} \right| \geq \frac{1}{256}$$

and

$$|3F_{111}^2 - F_{11}F_{1111}| = \frac{M^6}{4\pi^2 I^2} (U - V)^2 (U^2 + V^2 + 4UV) \geq \frac{1}{2\pi},$$

we can take $C_2 = C_3 = 0.003$. Similarly,

$$\Delta = \frac{M^{12}}{4I^3 \pi^4} (U - V)^3 U^2 V^2 (U^2 + UV + V^2) \geq 2^{-17} = C_4.$$

Now, in view of (11)–(13), by Lemma 6, following Huxley ([4], Theorem 3; [6], Theorem 3), we obtain that, for $0.3 \leq \beta < 0.407$,

$$|S(M)|^2 \ll T^{\frac{64}{205} + \beta}.$$

Now we consider the case $0.407 < \beta < 0.54$. Let $F(x)$ be a function four times continuously differentiable for $1 \leq x \leq 2$ with derivatives satisfying the following conditions:

$$|F^{(r)}(x)| \leq c_1 \tag{14}$$

for $r = 2, 3, 4$,

$$|F^{(r)}(x)| \geq 1/c_1 \quad (15)$$

for $r = 3, 4$, where C_1 is a positive constant, and

$$\left| F''(x)F^{(4)}(x) - 3(F^{(3)}(x))^2 \right| \geq c_3 \quad (16)$$

for some positive constant c_3 . Then by Proposition 1 and Theorem 1 of Huxley [5] we have that, for $0.407 < \beta \leq 0.5$ and $\varepsilon > 0$,

$$S(T^\beta) = \sum_{T^\alpha}^{M_2} e\left(T F\left(\frac{m}{M}\right)\right) \ll T^{\frac{32}{205} + \frac{\beta}{2} + \varepsilon},$$

where the implied constant depends only on c_1, c_3 , and ε .

In our case,

$$F(x) = \frac{\lambda M x}{T} - \frac{\log(Mx + \alpha)}{2\pi}.$$

Thus, by

$$\left| F''(x)F^{(4)}(x) - 3(F^{(3)}(x))^2 \right| = \frac{6}{(2\pi)^6(x + \alpha/M)^6},$$

we can choose $c_1 = 30$ and $c_3 = 10^{-7}$. Theorem 3 is proved.

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