On the zero distributions of Lerch zeta-functions

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Abstract. We study the zero distributions of the Lerch zeta-functions $L(\lambda, \alpha, s) = \sum_{n=0}^{\infty} \frac{e(\lambda n)}{(n+\alpha)^s}$ for the parameters $0 < \lambda, \alpha \leq 1$. Our observations show some analogies to the Riemann zeta-function (existence and number of trivial and nontrivial zeros) and some differences (asymmetrical distribution of the nontrivial zeros for almost all $L(\lambda, \alpha, s)$). Further, we investigate the distribution of zeros of the derivatives.

1 Introduction

As usual, let $s = \sigma + it$, $e(z) = \exp(2\pi iz)$. Denote by $\{\lambda\}$ the fractional part, and by $[\lambda]$ the integral part of a real number λ ; so $\lambda = [\lambda] + \{\lambda\}$.

Many problems in number theory are connected with the location of the zeros of the Riemann zeta-function, which is defined for $\sigma > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Riemann showed, that $\zeta(s)$ can be continued analytically to the whole complex plane except for a simple pole at $s = 1$, and satisfies a certain functional equation which forces the non-real zeros to be distributed symmetrically with respect to the so-called *critical line* $\sigma = \frac{1}{2}$ and the real axis. The yet unproved Riemann hypothesis states, that all non-real zeros of $\zeta(s)$ lie on the critical line. (See [8] and [12] for further reading). For an understanding of the analytical properties (Dirichlet series, analytic continuation, functional equation) and its consequences on the zero distribution it is important to study generalizations of the Riemann zeta-function.

For $0 < \alpha \leq 1, \lambda \in \mathbb{R}$, the Lerch zeta-function is given by

$$(1) \quad L(\lambda, \alpha, s) = \sum_{n=0}^{\infty} \frac{e(\lambda n)}{(n + \alpha)^s}.$$

This series converges absolutely for $\sigma > 1$. Obviously, we have $L(\lambda + k, \alpha, s) = L(\lambda, \alpha, s)$ for $k \in \mathbb{Z}$, so in the sequel we may assume $0 < \lambda \leq 1$. The analytic properties of $L(\lambda, \alpha, s)$ are quite different depending on $0 < \lambda < 1$ or $\lambda = 1$.

First, if $0 < \lambda < 1$, then the series (1) converges even for $\sigma > 0$. Moreover, one can prove a certain "functional equation"

$$\begin{aligned} L(\lambda, \alpha, 1-s) &= (2\pi)^{-s} \Gamma(s) \left(e\left(\frac{s}{4} - \alpha\lambda\right) L(-\alpha, \lambda, s) \right. \\ &\quad \left. + e\left(-\frac{s}{4} + \alpha(1-\lambda)\right) L(\alpha, 1-\lambda, s) \right). \end{aligned}$$

Hence $L(\lambda, \alpha, s)$ is an entire function.

For $\lambda = 1$ the Lerch zeta-function becomes the Hurwitz zeta-function $\zeta(s, \alpha) := L(1, \alpha, s)$, which can be continued analytically to the whole complex plane except for a simple pole at $s = 1$, and satisfies the "functional equation"

$$\zeta(1-s, \alpha) = 2(2\pi)^{-s} \Gamma(s) \left(\sin \frac{\pi(1-s)}{2} \sum_{n=1}^{\infty} \frac{\cos 2\pi n\alpha}{n^s} + \cos \frac{\pi(1-s)}{2} \sum_{n=1}^{\infty} \frac{\sin 2\pi n\alpha}{n^s} \right).$$

We can rewrite this in terms of Lerch zeta-functions. Then, introducing the notation $\{\lambda\}$, we can unify both functional equations by

$$(2) \quad \begin{aligned} L(\lambda, \alpha, 1-s) &= (2\pi)^{-s} \Gamma(s) \left(e\left(\frac{s}{4} - \alpha\lambda\right) L(-\alpha, \lambda, s) \right. \\ &\quad \left. + e\left(-\frac{s}{4} + \alpha(1-\{\lambda\})\right) L(\alpha, 1-\{\lambda\}, s) \right). \end{aligned}$$

$\zeta(s, \alpha)$ was introduced by Hurwitz [7], $L(\lambda, \alpha, s)$ by Lipschitz [6] and Lerch [4] (independently). Hurwitz and Lerch also gave the first proofs of the corresponding functional equations.

Note that $L\left(\frac{1}{2}, 1, s\right) = (1 - 2^{1-s})\zeta(s)$ has infinitely many zeros off the critical line $\sigma = \frac{1}{2}$, at least the zeros $s = 1 - \frac{2\pi i}{\log 2}k, k \in \mathbb{Z}$, of the factor $1 - 2^{1-s}$. So an analogue of the Riemann hypothesis could not hold for Lerch zeta-functions in general. As it was proved by Davenport and Heilbronn $\zeta(s, \alpha)$ has even infinitely many zeros in $1 < \sigma < 1 + \alpha$ if α is transcendental or rational $\neq \frac{1}{2}, 1$. This argument was extended by Cassels for α algebraic irrational. Moreover, in [1], [2], [3] those results are expanded

to Lerch zeta-functions, giving also lower and upper bounds for the number of zeros in various regions.

As in the case of the Riemann zeta-function we distinguish between so-called *trivial* and *nontrivial* zeros. The trivial zeros arise more or less directly from the functional equation and lie next to a straight line in the left half of the complex plane (see Section 3 for a rigorous definition).

In this paper we investigate the distribution of the nontrivial zeros of $L(\lambda, \alpha, s)$. We state our main results in the following section. Section 3 contains information concerning zero-free regions and the distribution of the *trivial* zeros, which we require in Section 4 to prove the main theorem. Finally, we discuss in Section 5 the zero distribution of the derivative of $L(\lambda, \alpha, s)$.

2 Statement of the main results

Let $\varrho = \beta + i\gamma$ denote the nontrivial zeros of $L(\lambda, \alpha, s)$. Let $N(T, \lambda, \alpha)$ count the number of nontrivial zeros ϱ of $L(\lambda, \alpha, s)$ with $|\gamma| \leq T$ (according multiplicities).

We will prove

Theorem 1 *Let $b \geq 3$ be a constant. Then we have for $0 < \lambda, \alpha \leq 1$, as T turns to infinity,*

$$\sum_{|\gamma| \leq T} (b + \beta) = \left(b + \frac{1}{2}\right) \frac{T}{\pi} \log \frac{T}{2\pi e \alpha \sqrt{\lambda(1 - \{\lambda\})}} + \frac{T}{2\pi} \log \frac{\alpha}{\sqrt{\lambda(1 - \{\lambda\})}} + O(\log T).$$

Using this with $b + 1$ instead of b , we get after subtracting the resulting formula from the one above

Corollary 2 *For $0 < \lambda, \alpha \leq 1$ we have, as T turns to infinity,*

$$N(T, \lambda, \alpha) = \frac{T}{\pi} \log \frac{T}{2\pi e \alpha \sqrt{\lambda(1 - \{\lambda\})}} + O(\log T).$$

This was first proved by Garunkštis and Laurinčikas [3].

Multiplying the formula of Corollary 2 with $b + \frac{1}{2}$ and subtracting it from the formula of Theorem 1 gives

Corollary 3 For $0 < \lambda, \alpha \leq 1$ we have, as T turns to infinity,

$$\sum_{|\gamma| \leq T} \left(\beta - \frac{1}{2} \right) = \frac{T}{2\pi} \log \frac{\alpha}{\sqrt{\lambda(1 - \{\lambda\})}} + O(\log T).$$

Hence we have $\sum_{|\gamma| \leq T} \left(\beta - \frac{1}{2} \right) = o(N(T, \lambda, \alpha))$.

Moreover, it follows that $L(\lambda, \alpha, s)$ has infinitely many zeros off the critical line if $\alpha^2 \neq \lambda(1 - \{\lambda\})$. Define

$$\Sigma(\lambda, \alpha) = \lim_{T \rightarrow \infty} \frac{2\pi}{T} \sum_{|\gamma| \leq T} \left(\beta - \frac{1}{2} \right).$$

Then, by Corollary 3, a non-zero $\Sigma(\lambda, \alpha)$ indicates an asymmetrical distribution of the nontrivial zeros of $L(\lambda, \alpha, s)$ (with respect to the critical line). Since

$$\int \int_{\substack{0 < \lambda, \alpha \leq 1 \\ \Sigma(\lambda, \alpha) < 0}} d\lambda d\alpha = \int_0^1 \sqrt{\lambda(1 - \lambda)} d\lambda = \frac{\pi}{8},$$

we get

Corollary 4 Almost all Lerch zeta-functions have an asymmetrical zero distribution. The measure of the number of the Lerch zeta-functions $L(\lambda, \alpha, s)$ with negative $\Sigma(\lambda, \alpha)$ is $\frac{\pi}{8} = 0.392\dots$, and the measure of those $L(\lambda, \alpha, s)$ with positive $\Sigma(\lambda, \alpha)$ is $1 - \frac{\pi}{8} = 0.607\dots$.

In the example $L\left(\frac{1}{2}, 1, s\right) = (1 - 2^{1-s})\zeta(s)$, where $\Sigma(\lambda, \alpha) = \log 2$, almost all zeros are distributed symmetrically, but infinitely many zeros are located on the right of $\sigma = \frac{1}{2}$. We conjecture that the non-trivial zeros of $L(\lambda, \alpha, s)$ are symmetrically distributed iff $\sum_{|\gamma| \leq T} \left(\beta - \frac{1}{2} \right)$ vanishes identically; moreover, we believe that this happens only for $L(1, 1, s) = \zeta(s)$ and $L\left(\frac{1}{2}, \frac{1}{2}, s\right) = 2^s L(s, \chi)$, where $L(s, \chi)$ is the Dirichlet L -function to the non-principal character $\chi \pmod{4}$.

Surprisingly, the asymmetries vanish if we look at all Lerch zeta-functions:

$$\int \int_{0 < \lambda, \alpha \leq 1} \Sigma(\lambda, \alpha) d\lambda d\alpha = 0.$$

3 Zero-free regions and trivial zeros

First, we require an approximation of $L(\lambda, \alpha, s)$ on the right:

Lemma 5 *For $\sigma > 2$ we have*

$$\alpha^{-\sigma} \left(1 - \frac{\alpha}{\sigma - 1}\right) \leq |L(\lambda, \alpha, s)| \leq \alpha^{-\sigma} \left(1 + \frac{\alpha}{\sigma - 1}\right).$$

Proof. For $\sigma > 1$ we have

$$|L(\lambda, \alpha, s) - \alpha^{-s}| \leq \sum_{n=1}^{\infty} \frac{1}{(n + \alpha)^{\sigma}} \leq \int_0^{\infty} \frac{dx}{(x + \alpha)^{\sigma}} = \frac{\alpha^{1-\sigma}}{\sigma - 1}.$$

This proves the Lemma. •

Moreover, this gives a zero-free region:

$$(3) \quad L(\lambda, \alpha, s) \neq 0 \quad \text{for} \quad \sigma > 1 + \alpha.$$

The situation on the left is more complicated. First, assume that $\lambda \neq \frac{1}{2}, 1$. Let l denote the line in the complex plane given by

$$\sigma = 1 + \frac{\pi t}{\log \frac{1-\lambda}{\lambda}}.$$

Let $\varrho(s, l)$ be the distance of s from l , and, for $\varepsilon > 0$, let

$$L_{\varepsilon}(l) = \{s \in \mathbb{C} : \varrho(s, l) < \varepsilon\}.$$

Further, define for $k \in \mathbb{Z}$

$$\sigma_k = 1 - \frac{2\pi(\alpha + k)}{\pi + \frac{1}{\pi} \left(\log \frac{1-\lambda}{\lambda}\right)^2}.$$

Then we have

Lemma 6 *Suppose $\lambda \neq \frac{1}{2}, 1$. There exist constants $\delta_0 \leq 0$ and $\varepsilon = \varepsilon(\sigma) > 0$, where $\varepsilon \rightarrow 0$ as $\sigma \rightarrow -\infty$, such that $L(\lambda, \alpha, s) \neq 0$ for $\sigma < \sigma_0$ and $s \notin L_{\varepsilon}(l)$. Moreover there exists a constant $\delta_1 \leq 0$ such that $L(\lambda, \alpha, s)$ has exactly one zero with real part between σ_k and σ_{k+1} , for $\sigma_k \leq \delta_1$.*

This follows with minor changes from the proofs of Theorem 2 and 3 in [3], which are based on the functional equation (2), the Dirichlet series representation (1) and, for the zero detection, Rouché's theorem.

For $\lambda = \frac{1}{2}$ we have

Lemma 7 *If $|t| \geq 1$ and $\sigma < -\frac{1}{2}$, then $L\left(\frac{1}{2}, \alpha, s\right) \neq 0$. Moreover if $\sigma \leq -\left(2\alpha + 1 + 2\left[\frac{3}{4} - \alpha\right]\right)$ and $|t| \leq 1$, then $L\left(\frac{1}{2}, \alpha, s\right) \neq 0$, except for zeros on the negative real axis, one in each interval $(-2n - 2\alpha - 1, -2n - 2\alpha + 1)$, $n \in \mathbb{N}$, $n \geq \frac{3}{4} - \alpha$.*

This follows from Theorem 4 and 5 in [3].

Note, that in all cases the number of trivial zeros with imaginary part in $[-T, T]$ is $\asymp T = o(N(T, \lambda, \alpha))$.

For $\lambda = 1$ Spira [10] proved

Lemma 8 *If $|t| \geq 1$ and $\sigma \leq -1$, then $\zeta(s, \alpha) \neq 0$. If $\sigma \leq -(4\alpha + 1 + 2[1 - 2\alpha])$ and $|t| \leq 1$, then $\zeta(s, \alpha) \neq 0$ except for zeros on the negative real axis, one in each interval $(-2n - 4\alpha - 1, -2n - 4\alpha + 1)$, $n \in \mathbb{Z}$, $n \geq 1 - 2\alpha$.*

A zero s_0 of $L(\lambda, \alpha, s)$ is called *trivial* if $s_0 \in L_\varepsilon(l)$ for $\lambda \neq \frac{1}{2}, 1$ (with ε given in Lemma 6), or s_0 lies on the real axis if $\lambda = \frac{1}{2}$ or $\lambda = 1$. Other zeros are called *nontrivial*.

4 Proof of Theorem 1

The proof follows a method due to Levinson and Montgomery [5] and makes use of Littlewood's Lemma, an integrated argument principle.

Define

$$(4) \quad Z(s) = \alpha^s L(\lambda, \alpha, s) = 1 + \sum_{n=1}^{\infty} e(\lambda n) \left(\frac{\alpha}{n + \alpha} \right)^s.$$

Note that the zeros of $Z(s)$ are exactly the zeros of $L(\lambda, \alpha, s)$. Let $a, b \geq 3$ be constants such that all nontrivial zeros of $L(\lambda, \alpha, s)$ have real parts in $(-b, a)$ (the existence of such constants follows from Lemma 6, 7, 8 and (3)). Let $N(\sigma, T)$ denote the number of nontrivial zeros ϱ of $L(\lambda, \alpha, s)$ with $\beta > \sigma$ and $|\gamma| \leq T$. Then Littlewood's Lemma (see [12], §9.9), applied to $Z(s)$ and the rectangle $\mathcal{R}$ with vertices $a \pm iT, -b \pm iT$, states

$$\int_{\mathcal{R}} \log Z(s) ds = -2\pi i \int_{-b}^a N(\sigma, T) d\sigma.$$

Hence

$$2\pi \sum_{|\gamma| \leq T} (b + \beta) = \int_{-T}^T \log |Z(-b + it)| dt - \int_{-T}^T \log |Z(a + it)| dt +$$

$$\begin{aligned}
& - \int_{-b}^a \arg Z(\sigma - iT) d\sigma + \int_{-b}^a \arg Z(\sigma + iT) d\sigma \\
& = \sum_{j=1}^4 I_j,
\end{aligned}$$

say. To define $\log Z(s)$ we choose the principal branch of the logarithm on the real axis, as $\sigma \rightarrow \infty$; for other points s the value of the logarithm is obtained by analytic continuation.

To evaluate I_1 note that

$$\log |Z(-b + it)| = b \log \frac{1}{\alpha} + \log |L(\lambda, \alpha, -b + it)|.$$

By the functional equation (2) we have

$$\begin{aligned}
(5) \quad & \log |L(\lambda, \alpha, -b + it)| = \log \left| (2\pi)^{-b-1+it} \Gamma(b+1-it) \right| + \\
& + \log \left| \exp \left(2\pi i \left(\frac{b+1}{4} - \alpha\lambda \right) + \frac{\pi t}{2} \right) L(-\alpha, \lambda, b+1-it) + \right. \\
& \left. + \exp \left(2\pi i \left(-\frac{b+1}{4} + \alpha(1 - \{\lambda\}) \right) - \frac{\pi t}{2} \right) L(\alpha, 1 - \{\lambda\}, b+1-it) \right|.
\end{aligned}$$

With Stirling's formula,

$$(6) \quad \log \Gamma(s) = \left(s - \frac{1}{2} \right) \log s - s + \frac{1}{2} \log 2\pi + O\left(\frac{1}{|s|} \right) \quad (|\arg s| \leq \pi - \varepsilon),$$

we get for $|t| > 1$

$$\log \left| (2\pi)^{-b-1+it} \Gamma(b+1-it) \right| = \left(b + \frac{1}{2} \right) \log \frac{|t|}{2\pi} - \frac{\pi|t|}{2} + O\left(\frac{1}{|t|} \right).$$

By Lemma 5 the second term on the right side of (5) equals, for $t > 1$,

$$\begin{aligned}
& \log \left| \exp \left(2\pi i \left(\frac{b+1}{4} - \alpha\lambda \right) + \frac{\pi t}{2} \right) L(-\alpha, \lambda, b+1-it) \times \right. \\
& \quad \times \left(1 + \exp \left(2\pi i \left(-\frac{b+1}{2} + \alpha(1 - \{\lambda\} + \lambda) \right) - \pi|t| \right) \times \right. \\
& \quad \left. \left. \times \frac{L(\alpha, 1 - \{\lambda\}, b+1-it)}{L(-\alpha, \lambda, b+1-it)} \right) \right| \\
& = \frac{\pi t}{2} + (b+1) \log \frac{1}{\lambda} + \log \left| \lambda^{b+1-it} L(-\alpha, \lambda, b+1-it) \right| + O(\exp(-\pi|t|)),
\end{aligned}$$

and, for $t < -1$,

$$-\frac{\pi t}{2} + (b+1) \log \frac{1}{1-\{\lambda\}} + \log \left| (1-\{\lambda\})^{b+1-it} L(\alpha, 1-\{\lambda\}, b+1-it) \right| \\ + O(\exp(-\pi|t|)).$$

Collecting together, we obtain

$$I_1 = 2 \left(b + \frac{1}{2} \right) T \log \frac{T}{2\pi e} + T \left((b+1) \log \frac{1}{\lambda(1-\{\lambda\})} + 2b \log \frac{1}{\alpha} \right) + \\ + \int_{-T}^T \log \left| \lambda^{b+1-it} L(-\alpha, \lambda, b+1-it) \right| dt + \\ + \int_{-T}^T \log \left| (1-\{\lambda\})^{b+1+it} L(\alpha, 1-\{\lambda\}, b+1+it) \right| dt + O(1).$$

The integrals above look similar to I_2 . We estimate them all as follows: for example in the case of I_2 , we have

$$I_2 = \int_{-T}^T \log \left| 1 + \sum_{n=1}^{\infty} e(\lambda n) \left(\frac{\alpha}{n+\alpha} \right)^{a+it} \right| dt.$$

Obviously, by Lemma 5, the modulus of the sum in I_2 is less than 1. Therefore we have

$$I_2 = \int_{-T}^T \operatorname{Re} \left(\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \left(\sum_{n=1}^{\infty} e(\lambda n) \left(\frac{\alpha}{n+\alpha} \right)^{a+it} \right)^m \right) dt \\ = \operatorname{Re} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \sum_{n_1=1}^{\infty} \dots \sum_{n_m=1}^{\infty} e(\lambda(n_1 + \dots + n_m)) \times \\ \times \left(\frac{\alpha^m}{(n_1+\alpha) \dots (n_m+\alpha)} \right)^a \int_{-T}^T \left(\frac{\alpha}{(n_1+\alpha) \dots (n_m+\alpha)} \right)^{it} dt \\ = O \left(\sum_{m=1}^{\infty} \frac{1}{m} \left(\sum_{n=1}^{\infty} \left(\frac{\alpha}{n+\alpha} \right)^a \right)^m \right).$$

Since $a \geq 3$ this is bounded.

Hence, we get at all

$$I_1 + I_2 = 2 \left(b + \frac{1}{2} \right) T \log \frac{T}{2\pi e \alpha \sqrt{\lambda(1-\{\lambda\})}} + T \log \frac{\alpha}{\sqrt{\lambda(1-\{\lambda\})}} + O(1).$$

It remains to estimate the horizontal integrals I_3, I_4 . Suppose that $\operatorname{Re} Z(\sigma + iT)$ has q zeros for $-b \leq \sigma \leq a$. Then divide $[-b, a]$ into at most $q+1$ subintervals in each of which $\operatorname{Re} Z(\sigma + iT)$ is of constant sign. Then

$$(7) \quad |\arg Z(\sigma + iT)| \leq (q+1)\pi.$$

To estimate q let

$$f(z) = \frac{1}{2} \left(Z(z + iT) + \overline{Z(\bar{z} + iT)} \right).$$

Then, we have $f(\sigma) = \operatorname{Re} Z(\sigma + iT)$. Let $R = a + b$ and choose T so large that $T > 2R$. Now, $\operatorname{Im}(z + iT) > 0$ for $|z - a| < T$. Thus $Z(z + iT)$, and hence $f(z)$, is analytic for $|z - a| < T$. Let $n(r)$ denote the number of zeros of $f(z)$ in $|z - a| \leq r$. Obviously, we have

$$\int_0^{2R} \frac{n(r)}{r} dr \geq n(R) \int_R^{2R} \frac{dr}{r} = n(R) \log 2.$$

With Jensen's formula ([11], §3.61),

$$\int_0^{2R} \frac{n(r)}{r} dr = \frac{1}{2\pi} \int_0^{2\pi} \log |f(a + 2Re^{i\theta})| d\theta - \log |f(a)|,$$

we deduce

$$n(R) \leq \frac{1}{2\pi \log 2} \int_0^{2\pi} \log |f(a + 2Re^{i\theta})| d\theta - \frac{\log |f(a)|}{\log 2}.$$

By Lemma 5 it follows that $\log |f(a)|$ is bounded. To bound the integrand above, note that we have by Stirling's formula (6), the functional equation (2) and Lemma 5

$$L(\lambda, \alpha, 1 - s) = O(|t|^{\sigma - \frac{1}{2}}).$$

Using the Phragmén-Lindelöf principle ([11], §5.65), we get in any bounded strip

$$L(\lambda, \alpha, s) = O(|t|^c)$$

with a certain constant $c > 0$. Obviously, the same estimate holds for $f(s)$. Thus, the integral above is $O(\log T)$, and $n(R) = O(\log T)$. Since the interval $(-b, a)$ is contained in the disc $|z - a| \leq R$, we have $q \leq n(R)$. Therefore, with (7), we get

$$I_4 \leq \int_{-b}^a |\arg Z(\sigma + iT)| d\sigma = O(\log T).$$

Obviously, I_3 can be bounded in the same way. Thus we have proved Theorem 1. •

5 Further results for the derivative

Define

$$L'(\lambda, \alpha, s) = \frac{\partial}{\partial s} L(\lambda, \alpha, s).$$

One can show by standard arguments that $L'(\lambda, \alpha, s)$ has similar zero-free regions as $L(\lambda, \alpha, s)$, so we can speak once more of trivial and nontrivial zeros of $L'(\lambda, \alpha, s)$ in the above sense. Denote by $\varrho' = \beta' + i\gamma'$ the nontrivial zeros of $L(\lambda, \alpha, s)$. Let $N'(T, \lambda, \alpha)$ count the number of nontrivial zeros ϱ' of $L'(\lambda, \alpha, s)$ with $|\gamma'| \leq T$ (according multiplicities).

A slight modification in the proof of Theorem 1 generalizes results of Levinson and Montgomery [5] on the Riemann zeta-function and its derivative. We argue with

$$-\frac{([\alpha] + \alpha)^s}{\log([\alpha] + \alpha)} L'(\lambda, \alpha, s) = 1 + \sum_{n=[\alpha]+1}^{\infty} e(\lambda n) \frac{\log(n + \alpha)}{\log([\alpha] + \alpha)} \left(\frac{[\alpha] + \alpha}{n + \alpha} \right)^s$$

instead of (4). Proceeding as above, the contribution of the first factor for I_1 is

$$2T \left(b \log \frac{1}{[\alpha] + \alpha} - \log \log \frac{1}{[\alpha] + \alpha} \right).$$

Differentiation of the functional equation (2) yields

$$\begin{aligned} -L'(\lambda, \alpha, 1-s) &= (2\pi)^{-s} \Gamma(s) \times \left(e \left(\frac{s}{4} - \alpha \lambda \right) (L'(-\alpha, \lambda, s) + \right. \\ (8) \quad &+ L(-\alpha, \lambda, s) \left(\frac{\Gamma'}{\Gamma}(s) - \log 2\pi + \frac{\pi i}{2} \right)) + \\ &+ e \left(-\frac{s}{4} + \alpha(1 - \{\lambda\}) \right) (L'(\alpha, 1 - \{\lambda\}, s) + \\ &\left. + L(\alpha, 1 - \{\lambda\}, s) \left(\frac{\Gamma'}{\Gamma}(s) - \log 2\pi - \frac{\pi i}{2} \right) \right). \end{aligned}$$

By Stirling's formula (6), we get for $|t| > 1$

$$(9) \quad \frac{\Gamma'}{\Gamma}(s) - \log 2\pi \pm \frac{\pi i}{2} = \log \left(\frac{|t|}{2\pi} \exp \left(\pm \frac{\pi i}{2} \right) \right) + O \left(\frac{1}{|t|} \right).$$

Therefore we have in I_1 the additional contribution

$$\begin{aligned} &\int_0^T \left(\log \log \left(\frac{t}{2\pi} \exp \left(\frac{\pi i}{2} \right) \right) + \log \log \left(\frac{t}{2\pi} \exp \left(-\frac{\pi i}{2} \right) \right) \right) dt \\ &= T \log \left(\left(\log \frac{T}{2\pi} \right)^2 + \frac{\pi}{2} \right) + o(T). \end{aligned}$$

The other integrals can be estimated as before. Thus we get at all

$$\begin{aligned} \sum_{|\gamma'| \leq T} (b + \beta') &= \left(b + \frac{1}{2}\right) \frac{T}{\pi} \log \frac{T}{2\pi e([\alpha] + \alpha) \sqrt{\lambda(1 - \{\lambda\})}} + \\ &+ \frac{T}{2\pi} \log \frac{[\alpha] + \alpha}{\sqrt{\lambda(1 - \{\lambda\})}} - \frac{T}{\pi} \log \log \frac{1}{[\alpha] + \alpha} + \\ &+ \frac{T}{2\pi} \log \left(\left(\log \frac{T}{2\pi} \right)^2 + \frac{\pi}{2} \right) + o(T). \end{aligned}$$

From this we deduce

$$N'(T, \lambda, \alpha) = \frac{T}{\pi} \log \frac{T}{2\pi e([\alpha] + \alpha) \sqrt{\lambda(1 - \{\lambda\})}} + o(T).$$

Thus, we have $N(T, \lambda, \alpha) - N'(T, \lambda, \alpha) = o(T)$ whenever $\alpha \neq 1$.

Moreover we get, independent on the sign of $\Sigma(\lambda, \alpha)$,

$$\sum_{|\gamma'| \leq T} \left(\beta' - \frac{1}{2} \right) = \frac{T}{\pi} \log \log T + O(T).$$

Speiser [9] showed that Riemann's hypothesis is equivalent to the non-vanishing of $\zeta'(s)$ in $-1 < \sigma < \frac{1}{2}$. Now we will show that $L'(\lambda, \alpha, s)$ with irrational λ has more than cT zeros in any rectangle $-\varepsilon < \sigma < 0, -T \leq t \leq T$, where c is a positive constant, almost depending on $\varepsilon > 0$.

Suppose $\sigma > 1 + \varepsilon$. Using Dirichlet's approximation theorem ([12], §8.2) one can prove that for every $\varepsilon > 0$, the interval $[-T, T]$ contains more than cT numbers τ with

$$|L(-\alpha, \lambda, s + i\tau) - L(-\alpha, \lambda, s)| < \varepsilon$$

(for more details see the proof of Lemma 1 in [1]). If λ is irrational, then from Theorem 1 in [1] follows that $L(-\alpha, \lambda, s)$ has a zero s_0 in any strip $1 < \sigma < 1 + \varepsilon$ (s_0 depends from ε). Now in this strip we choose a circle with center s_0 , such that $L(-\alpha, \lambda, s)$ does not vanish on the boundary. Obviously, $L'(-\alpha, \lambda, s)$ is bounded on the circle. Let τ be positive and sufficiently large. In view of the above and (9) we can apply Rouché's theorem for the functions $L(-\alpha, \lambda, s)$ and

$$L(-\alpha, \lambda, s) + \left(L(-\alpha, \lambda, s + i\tau) - L(-\alpha, \lambda, s) + \frac{L'(-\alpha, \lambda, s + i\tau)}{\frac{\Gamma'}{\Gamma}(s + i\tau) - \log 2\pi + \frac{\pi i}{2}} + \right.$$

$$\begin{aligned}
& + \frac{e\left(-\frac{s+i\tau}{2} + \alpha\right)}{\frac{\Gamma'}{\Gamma}(s+i\tau) - \log 2\pi + \frac{\pi i}{2}} \left(L'(\alpha, 1-\lambda, s+i\tau) + \right. \\
& \left. + L(\alpha, 1-\lambda, s+i\tau) \left(\frac{\Gamma'}{\Gamma}(s+i\tau) - \log 2\pi - \frac{\pi i}{2} \right) \right)
\end{aligned}$$

For τ negative and sufficiently large we argue similarly. By (8) we deduce that $L'(\lambda, \alpha, s)$ has more than cT zeros in any rectangle $-\varepsilon < \sigma < 0$, $-T \leq t \leq T$.

By the way, this gives a correspondence between certain zeros of $L(-\alpha, \lambda, s)$ in $\sigma > 1$ and the zeros of $L'(\lambda, \alpha, s)$ in $\sigma < 0$. Maybe, this is the analogue of Levinson and Montgomery's quantitative version (Theorem 1 in [5]) of Speiser's correspondence, namely that there are as many nontrivial zeros of $\zeta(s)$ on the right of the critical line as nontrivial zeros of $\zeta'(s)$ on the left (apart from a small hypothetical error).

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