On the mean square of Lerch zeta-functions

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Abstract. Using the approximate functional equation for Lerch zeta-functions $L(\lambda, \alpha, s) = \sum_{n=0}^{\infty} \frac{e(\lambda n)}{(n+\alpha)^s}$, we prove for fixed parameters $0 < \lambda, \alpha \leq 1$ asymptotic formulas for the mean square of $L(\lambda, \alpha, s)$ inside the critical strip. This improves earlier results of D. Klusch and of A. Laurinćikas.

Keywords: Lerch zeta-function, mean square.

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1 Introduction and statement of results

As usual let $s = \sigma + it$ and $e(z) = \exp(2\pi iz)$. Further, denote by $[\lambda]$ the integral and by $\{\lambda\}$ the fractional part of a real number λ . For the parameters $0 < \alpha \leq 1, \lambda \in \mathbb{R}$ the Lerch zeta-function is given by

$$L(\lambda, \alpha, s) = \sum_{n=0}^{\infty} \frac{e(\lambda n)}{(n+\alpha)^s} \quad (\sigma > 1).$$

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Obviously $L(\lambda, \alpha, s) = L(\{\lambda\}, \alpha, s)$, so in the sequel we may assume $0 < \lambda \leq 1$. Lerch zeta-functions have an analytical continuation to the whole complex plane except for at most one simple pole at $s = 1$. $L(\lambda, \alpha, s)$ is irregular at $s = 1$ if and only if $\lambda \in \mathbb{Z}$, in which case the Lerch zeta-function becomes the so-called Hurwitz zeta-function $\zeta(s, \alpha) := L(1, \alpha, s)$; especially for $\lambda = \alpha = 1$ we obtain the well known Riemann zeta-function. In any case, Lerch zeta-functions satisfy the functional equation

$$(1) \quad L(\lambda, \alpha, 1-s) = \frac{\Gamma(s)}{(2\pi)^s} \left(e\left(\frac{s}{4} - \alpha\lambda\right) L(-\alpha, \lambda, s) + e\left(-\frac{s}{4} + \alpha(1 - \{\lambda\})\right) L(\alpha, 1 - \{\lambda\}, s) \right),$$

which was first proved by Lerch [10] for $0 < \lambda < 1$, and by Hurwitz [6] for $\lambda = 1$.

For analytical investigations it is often useful to deal with approximations. For example, if $0 \leq \sigma_0 \leq \sigma \leq 2$, $0 < |t| \leq \pi\lambda N$, then

$$(2) \quad L(\lambda, \alpha, s) = \sum_{0 \leq n \leq N} \frac{e(\lambda n)}{(n + \alpha)^s} + [\lambda] \frac{N^{1-s}}{s-1} + O(N^{-\sigma}).$$

A proof of (2) for $\lambda = 1$ and $\alpha = 1$, the case of the Riemann zeta-function, can be found in [7]; for $\lambda = 1$, $0 < \alpha \leq 1$, the case of Hurwitz zeta-function, follows with the same arguments; for Lerch zeta-function this formula was proved in [3]. In [4] the authors proved the following approximate functional equation (which combines the functional equation (1) and approximating Dirichlet polynomials like the one in (2)): if $0 < \lambda, \alpha \leq 1$ and $0 \leq \sigma \leq 1$ are fixed, then we have, for $t \geq 1$,

$$(3) \quad L(\lambda, \alpha, s) = \sum_{0 \leq n \leq m(t)} \frac{e(\lambda n)}{(n + \alpha)^s} + \left(\frac{t}{2\pi}\right)^{\frac{1}{2}-\sigma-it} e^{it+\frac{\pi i}{4}-2\pi i\{\lambda\}\alpha} \sum_{0 \leq n \leq q(t)} \frac{e(-\alpha n)}{(n + \lambda)^{1-s}} + \left(\frac{t}{2\pi}\right)^{-\frac{\sigma}{2}} e^{\pi i f(\lambda, \alpha, t)} \psi(g(\lambda, \alpha, t)) + O(t^{\frac{\sigma}{2}-1}),$$

where

$$\begin{aligned} m(t) &:= \left\lfloor \sqrt{\frac{t}{2\pi}} - \alpha \right\rfloor, & q(t) &:= \left\lfloor \sqrt{\frac{t}{2\pi}} \right\rfloor, \\ g(\lambda, \alpha, t) &:= 2\sqrt{\frac{t}{2\pi}} - q(t) - m(t) - \{\lambda\} - \alpha, \\ f(\lambda, \alpha, t) &:= -\frac{t}{2\pi} \log \frac{t}{2\pi e} - \frac{7}{8} + \frac{1}{2}(\alpha^2 - \{\lambda\}^2) - \alpha(q(t) - m(t)) \\ &\quad + 2\sqrt{\frac{t}{2\pi}}(q(t) - m(t) + \{\lambda\} - \alpha) - \frac{1}{2}(q(t) + m(t)) \\ &\quad - \{\lambda\}(\alpha + (q(t) - m(t))) \end{aligned}$$

and

$$\psi(z) := \frac{\cos \pi \left(\frac{1}{2}z^2 - z - \frac{1}{8} \right)}{\cos \pi z}.$$

Note that $\psi(z)$ is everywhere regular. Lets define the constant $c(\alpha)$ by

$$\sum_{n=0}^M (n + \alpha)^{-1} = \log M + c(\alpha) + O(1/M)$$

as $M \rightarrow \infty$. Using the approximate functional equation we will prove

Theorem 1 *For $0 < \lambda, \alpha \leq 1$ we have, as T turns to infinity,*

$$\int_1^T |L(\lambda, \alpha, 1/2 + it)|^2 dt = T \log \frac{T}{2\pi} + T(c(\alpha) + c(\lambda) - 1) + O\left(T^{\frac{1}{2}} \log T\right)$$

and for $\frac{1}{2} < \sigma < 1$,

$$\begin{aligned} \int_1^T |L(\lambda, \alpha, \sigma + it)|^2 dt &= \zeta(2\sigma, \alpha)T + \frac{(2\pi)^{2\sigma-1}}{2-2\sigma} \zeta(2-2\sigma, \lambda)T^{2-2\sigma} \\ &\quad + O\left(T^{1-\sigma} \log T + T^{\frac{\sigma}{2}}\right). \end{aligned}$$

These formulas extend a result of Rane [12] (who proved a mean square formula for Hurwitz zeta-functions on $\sigma = \frac{1}{2}$ with the same error term), improves some results of Klusch [8] (mean square formulas for Lerch zeta-functions for $\frac{1}{2} \leq \sigma < 1$ without error term), also improves results from [9], where Theorem 1 by help of the approximation (2) was obtained with the error term $O(T^{2-2\sigma})$.

2 Preliminaries

In the proof of Theorem 1 we will often make use of

Lemma 2 *For $0 < \lambda, \alpha \leq 1$ and $\beta < 1, \beta + \gamma < 1$ we have*

$$\sum_{0 \leq m < n \leq Q} (m + \alpha)^{-\beta} (n + \lambda)^{-\gamma} \left(\log \frac{n + \lambda}{m + \alpha} \right)^{-1} \ll Q^{2-\beta-\gamma} \log Q.$$

Proof. We split the sum according to the cases $m < \frac{n}{2}$ and $\frac{n}{2} \leq m < n$. Obviously, we have in the first case

$$\begin{aligned} &\sum_{1 \leq n \leq Q} \sum_{m < \frac{n}{2}} (m + \alpha)^{-\beta} (n + \lambda)^{-\gamma} \left(\log \frac{n + \lambda}{m + \alpha} \right)^{-1} \\ &\ll \sum_{1 \leq n \leq Q} n^{-\gamma} \sum_{m < n} m^{-\beta} \ll \sum_{1 \leq n \leq Q} n^{1-\beta-\gamma} \ll Q^{2-\beta-\gamma}. \end{aligned}$$

If we write $m = n - r$ in the second case, then we get by the Taylor expansion of the logarithm $\log \frac{n+\alpha}{m+\lambda} \gg \frac{r}{n}$ for $1 \leq r < \frac{n}{2}$. Thus, we obtain

$$\begin{aligned} & \sum_{1 \leq n \leq Q} \sum_{\frac{n}{2} \leq m < n} (m + \alpha)^{-\beta} (n + \lambda)^{-\gamma} \left(\log \frac{n + \lambda}{m + \alpha} \right)^{-1} \\ & \ll \sum_{1 \leq n \leq Q} (n + \lambda)^{-\gamma} \sum_{1 \leq r < \frac{n}{2}} (n + \alpha - r)^{-\beta} \frac{n}{r} \\ & \ll \sum_{1 \leq n \leq Q} n^{1-\beta-\gamma} \sum_{1 \leq r \leq n} \frac{1}{r} \ll Q^{2-\beta-\gamma} \log Q. \end{aligned}$$

Collecting both cases together, the estimate of the lemma follows. •

Moreover, we need some estimates of exponential integrals.

Lemma 3 (First derivative test) *Let $g(t)$ be real and differentiable with monotonic $g'(t) \geq \delta_1 > 0$ and $G(t)$ monotonic with $|G(t)| \leq G$ on $[A, B]$. Then*

$$\int_A^B G(t) e^{ig(t)} dt \ll G \delta_1^{-1}.$$

Lemma 4 (Second derivative test) *Let $g(t)$ be real, twice differentiable with $g''(t) \geq \delta_2 > 0$ throughout $[A, B]$. Then*

$$\int_A^B e^{ig(t)} dt \ll \delta_2^{-\frac{1}{2}}.$$

Proofs can be found in [7]. Note that if $G(t)$ and $g(t)$ both are twice differentiable on $[A, B]$, then one can get for the integral in Lemma 3 by partial integration

$$(4) \quad \int_A^B G(t) e^{ig(t)} dt \ll \max_{A \leq t \leq B} \left| \frac{G(t)}{g'(t)} \right| + (B - A) \max_{A \leq t \leq B} \left| \frac{G'(t)g'(t) - G(t)g''(t)}{g'(t)^2} \right|.$$

3 Proof of Theorem 1

Using the approximate functional equation (3), we have

$$\begin{aligned} & \int_{\frac{T}{2}}^T |L(\lambda, \alpha, \sigma + it)|^2 dt \\ & = \int_{\frac{T}{2}}^T \sum_{0 \leq m, n \leq m(t)} \frac{e(\lambda(m - n))}{(m + \alpha)^{\sigma+it} (n + \alpha)^{\sigma-it}} dt \\ & \quad + \int_{\frac{T}{2}}^T \left(\frac{t}{2\pi} \right)^{1-2\sigma} \sum_{0 \leq m, n \leq q(t)} \frac{e(\alpha(m - n))}{(m + \lambda)^{1-\sigma-it} (n + \lambda)^{1-\sigma+it}} dt \end{aligned}$$

$$\begin{aligned}
& + \int_{\frac{T}{2}}^T \left(\frac{t}{2\pi} \right)^{-\sigma} \psi^2(g(\lambda, \alpha, t)) dt + O \left(\int_{\frac{T}{2}}^T t^{\sigma-2} dt \right) \\
& + 2\operatorname{Re} \left(\int_{\frac{T}{2}}^T \left(\frac{t}{2\pi} \right)^{\frac{1}{2}-\sigma+it} e^{-it-\frac{\pi i}{4}+2\pi i\{\lambda\}\alpha} \times \right. \\
& \times \sum_{0 \leq m, n \leq q(t)} \frac{e(\lambda m + \alpha n)}{(m + \alpha)^{\sigma+it}(n + \lambda)^{1-\sigma+it}} dt \\
& + \int_{\frac{T}{2}}^T \left(\frac{t}{2\pi} \right)^{-\frac{\sigma}{2}} e^{-\pi i f(\lambda, \alpha, s)} \psi(g(\lambda, \alpha, t)) \sum_{0 \leq n \leq m(t)} \frac{e(\lambda n)}{(n + \alpha)^{\sigma+it}} dt \\
& + \int_{\frac{T}{2}}^T \left(\frac{t}{2\pi} \right)^{\frac{1}{2}-\frac{3\sigma}{2}-it} e^{it+\frac{\pi i}{4}-2\pi i\{\lambda\}\alpha-\pi i f(\lambda, \alpha, s)} \psi(g(\lambda, \alpha, t)) \times \\
& \times \sum_{0 \leq n \leq q(t)} \frac{e(-\alpha n)}{(n + \lambda)^{1-\sigma-it}} dt + \int_{\frac{T}{2}}^T O \left(t^{\frac{\sigma}{2}-1} \right) \sum_{0 \leq n \leq m(t)} \frac{e(\lambda n)}{(n + \alpha)^{\sigma+it}} dt \\
& + \int_{\frac{T}{2}}^T O \left(t^{\frac{\sigma}{2}-1} \right) \left(\frac{t}{2\pi} \right)^{\frac{1}{2}-\sigma-it} e^{it+\frac{\pi i}{4}-2\pi i\{\lambda\}\alpha} \sum_{0 \leq n \leq q(t)} \frac{e(-\alpha n)}{(n + \lambda)^{1-\sigma-it}} dt \\
& + \int_{\frac{T}{2}}^T O \left(t^{\frac{\sigma}{2}-1} \right) \left(\frac{t}{2\pi} \right)^{-\frac{\sigma}{2}} e^{\pi i f(\lambda, \alpha, s)} \psi(g(\lambda, \alpha, t)) dt \Big) \\
& =: \sum_{1 \leq j \leq 10} \mathcal{I}_j.
\end{aligned}$$

First, we find

$$\mathcal{I}_1 = \sum_{0 \leq m, n \leq m(T)} \frac{e(\lambda(n-m))}{(m+\alpha)^\sigma(n+\alpha)^\sigma} \int_{T_1}^T \left(\frac{m+\alpha}{n+\alpha} \right)^{it} dt$$

with $T_1 := \max \left\{ \frac{T}{2}, 2\pi(m+\alpha)^2, 2\pi(n+\alpha)^2 \right\}$. The contribution of the diagonal terms is

$$\begin{aligned}
& \sum_{0 \leq n \leq m(T)} (n+\alpha)^{-2\sigma} \left(T - \max \left\{ \frac{T}{2}, 2\pi(n+\alpha)^2 \right\} \right) \\
& = T \sum_{0 \leq n \leq m(T)} (n+\alpha)^{-2\sigma} - \frac{T}{2} \sum_{0 \leq n \leq m(\frac{T}{2})} (n+\alpha)^{-2\sigma} \\
& \quad - 2\pi \sum_{m(\frac{T}{2})+1 \leq n \leq m(T)} (n+\alpha)^{2-2\sigma}.
\end{aligned}$$

for $\sigma = 1/2$ this is equal to

$$\frac{T}{4} \log \frac{T}{\pi} + \frac{T}{2} (c(\alpha) - \frac{1}{2}) + O(T^{1/2}).$$

For $1/2 < \sigma < 1$, by use of (2) the diagonal terms above equal

$$\frac{1}{2}\zeta(2\sigma, \alpha)T - \frac{2(2\pi)^{\sigma-\frac{1}{2}}(1-2^{\sigma-\frac{3}{2}})}{(2\sigma-1)(3-2\sigma)}T^{\frac{3}{2}-\sigma} + O(T^{1-\sigma}).$$

The non-diagonal terms are

$$\begin{aligned} & \sum_{\substack{0 \leq m, n \leq m(T) \\ m \neq n}} \frac{e(\lambda(n-m))}{(m+\alpha)^\sigma(n+\alpha)^\sigma} \int_{T_1}^T \exp\left(it \log \frac{m+\alpha}{n+\alpha}\right) dt \\ &= \sum_{\substack{0 \leq m, n \leq m(T) \\ m \neq n}} \frac{e(\lambda(n-m))}{(m+\alpha)^\sigma(n+\alpha)^\sigma} \left(i \log \frac{m+\alpha}{n+\alpha}\right)^{-1} \times \\ & \quad \times \left(\exp\left(iT \log \frac{m+\alpha}{n+\alpha}\right) - \exp\left(iT_1 \log \frac{m+\alpha}{n+\alpha}\right)\right) \\ &\ll \sum_{0 \leq m < n \leq m(T)} (m+\alpha)^{-\sigma}(n+\alpha)^{-\sigma} \left(\log \frac{n+\alpha}{m+\alpha}\right)^{-1}. \end{aligned}$$

Applying Lemma 2, the non-diagonal terms turn out to be $\ll T^{1-\sigma} \log T$. Hence, for $\sigma = 1/2$,

$$(5) \quad \mathcal{I}_1 = \frac{T}{4} \log \frac{T}{\pi} + \frac{T}{2} (c(\alpha) - \frac{1}{2}) + O(T^{1/2} \log T),$$

and, for $1/2 < \sigma < 1$,

$$(6) \quad \mathcal{I}_1 = \frac{1}{2}\zeta(2\sigma, \alpha)T - \frac{2(2\pi)^{\sigma-\frac{1}{2}}(1-2^{\sigma-\frac{3}{2}})}{(2\sigma-1)(3-2\sigma)}T^{\frac{3}{2}-\sigma} + O(T^{1-\sigma} \log T).$$

Now

$$\mathcal{I}_2 = \sum_{0 \leq m, n \leq q(T)} \frac{e(\alpha(m-n))}{(m+\lambda)^{1-\sigma}(n+\lambda)^{1-\sigma}} \int_{T_2}^T \left(\frac{t}{2\pi}\right)^{1-2\sigma} \exp\left(it \log \frac{n+\lambda}{m+\lambda}\right) dt$$

with $T_2 := \max\left\{\frac{T}{2}, 2\pi(m+\lambda)^2, 2\pi(n+\lambda)^2\right\}$. For the diagonal terms we get

$$\begin{aligned} & \frac{(2\pi)^{2\sigma-1}}{2-2\sigma} \sum_{0 \leq n \leq q(T)} (n+\lambda)^{2\sigma-2} \left(T^{2-2\sigma} - \max\left\{\frac{T}{2}, 2\pi(n+\lambda)^2\right\}^{2-2\sigma}\right) \\ &= \frac{(2\pi)^{2\sigma-1}}{2-2\sigma} \left(T^{2-2\sigma} \sum_{0 \leq n \leq q(T)} (n+\lambda)^{2\sigma-2} - \left(\frac{T}{2}\right)^{2-2\sigma} \sum_{0 \leq n \leq q(\frac{T}{2})} (n+\lambda)^{2\sigma-2} \right. \\ & \quad \left. - (2\pi)^{2-2\sigma} \sum_{q(\frac{T}{2})+1 \leq n \leq q(T)} (n+\lambda)^{2\sigma-2} + O(T^{1-\sigma})\right). \end{aligned}$$

For $\sigma = 1/2$ this is equal to

$$\frac{T}{4} \log \frac{T}{\pi} + \frac{T}{2} (c(\lambda) - \frac{1}{2}) + O(T^{1/2})$$

and for $1/2 < \sigma < 1$, to

$$\frac{(2\pi)^{2\sigma-1}(1-2^{2\sigma-2})}{2-2\sigma} \zeta(2-2\sigma, \lambda) T^{2-2\sigma} + \frac{2(2\pi)^{\sigma-\frac{1}{2}}(1-2^{\sigma-\frac{3}{2}})}{(2\sigma-1)(3-2\sigma)} T^{\frac{3}{2}-\sigma} + O(T^{1-\sigma}).$$

Lemma 3 yields for $m \neq n$

$$\int_{T_2}^T \left(\frac{t}{2\pi}\right)^{1-2\sigma} \exp\left(it \log \frac{n+\lambda}{m+\lambda}\right) dt \ll T^{1-2\sigma} \left|\log \frac{n+\lambda}{m+\lambda}\right|^{-1}.$$

Therefore, we find for the non-diagonal terms

$$\begin{aligned} & \sum_{\substack{0 \leq m, n \leq q(T) \\ m \neq n}} \frac{e(\alpha(m-n))}{(m+\lambda)^{1-\sigma}(n+\lambda)^{1-\sigma}} \int_{T_2}^T \left(\frac{t}{2\pi}\right)^{1-2\sigma} \exp\left(it \log \frac{n+\lambda}{m+\lambda}\right) dt \\ & \ll T^{1-2\sigma} \sum_{0 \leq m < n \leq q(T)} (m+\lambda)^{\sigma-1} (n+\lambda)^{\sigma-1} \left(\log \frac{n+\lambda}{m+\lambda}\right)^{-1}. \end{aligned}$$

By use of Lemma 2 this is $\ll T^{1-\sigma} \log T$. Thus,

$$\mathcal{I}_2 = \frac{T}{4} \log \frac{T}{\pi} + \frac{T}{2} (c(\lambda) - \frac{1}{2}) + O(T^{1/2} \log T), \quad \text{for } \sigma = 1/2$$

and

$$\begin{aligned} \mathcal{I}_2 &= \frac{(2\pi)^{2\sigma-1}(1-2^{2\sigma-2})}{2-2\sigma} \zeta(2-2\sigma, \lambda) T^{2-2\sigma} + \frac{2(2\pi)^{\sigma-\frac{1}{2}}(1-2^{\sigma-\frac{3}{2}})}{(2\sigma-1)(3-2\sigma)} T^{\frac{3}{2}-\sigma} \\ &+ O(T^{1-\sigma} \log T), \quad \text{for } 1/2 < \sigma < 1. \end{aligned}$$

Since the argument $g(\lambda, \alpha, t)$ of ψ in $\mathcal{I}_3$ is bounded, we get easily $\mathcal{I}_3 \ll T^{1-\sigma}$. Further, we have $\mathcal{I}_4 \ll T^{\sigma-1}$. Now we consider

$$\mathcal{I}_5 \ll \sum_{0 \leq m, n \leq q(T)} (m+\alpha)^{-\sigma} (n+\lambda)^{\sigma-1} \left| \int_{T_5}^T t^{\frac{1}{2}-\sigma} e^{if(t)} dt \right|$$

with $T_5 := \max \left\{ \frac{T}{2}, 2\pi(m+\alpha)^2, 2\pi(n+\lambda)^2 \right\}$ and

$$f(t) := t \log \frac{t}{2\pi e(m+\alpha)(n+\lambda)}.$$

Lemma 4 gives in combination with Lemma 3

$$(7) \quad \int_{T_5}^T e^{if(t)} dt \ll \begin{cases} T^{\frac{1}{2}}, \\ \left(\log \frac{\max\{(m+\alpha)^2, (n+\lambda)^2\}}{(m+\alpha)(n+\lambda)} \right)^{-1} \end{cases}.$$

The function $t^{\frac{1}{2}-\sigma}$ is decreasing and positive. Therefore, application of the second mean value theorem for the real and the imaginary part separately, leads with regard to (7) and $T \asymp T_5$ to

$$\int_{T_5}^T t^{\frac{1}{2}-\sigma} e^{if(t)} dt \ll T^{\frac{1}{2}-\sigma} \left| \int_{T_5}^T e^{if(t)} dt \right|.$$

This yields

$$\begin{aligned} \mathcal{I}_5 &\ll T^{\frac{1}{2}-\sigma} \sum_{\substack{0 \leq m, n \leq q(T) \\ |m-n| \geq 2}} (m+\alpha)^{-\sigma} (n+\lambda)^{\sigma-1} \left(\log \frac{\max\{(m+\alpha)^2, (n+\lambda)^2\}}{(m+\alpha)(n+\lambda)} \right)^{-1} \\ &\quad + T^{1-\sigma} \sum_{\substack{0 \leq m, n \leq q(T) \\ |m-n| \leq 1}} (m+\alpha)^{-\sigma} (n+\lambda)^{\sigma-1} \\ &\ll T^{\frac{1}{2}-\sigma} \left(\sum_{0 \leq m < n \leq q(T)} (m+\alpha)^{-\sigma} (n+\lambda)^{\sigma-1} \left(\log \frac{n+\lambda}{m+\alpha} \right)^{-1} \right. \\ &\quad \left. + \sum_{0 \leq n < m \leq q(T)} (m+\alpha)^{-\sigma} (n+\lambda)^{\sigma-1} \left(\log \frac{m+\alpha}{n+\lambda} \right)^{-1} \right) \\ &\quad + T^{1-\sigma} \left(1 + \sum_{1 \leq m \leq q(T)} m^{-1} \right). \end{aligned}$$

Using Lemma 2, we obtain once more $\mathcal{I}_5 \ll T^{1-\sigma} \log T$. Now

$$\begin{aligned} \mathcal{I}_6 &\ll \left| \int_{\frac{T}{2}}^T t^{-\frac{\sigma}{2}} e^{-\pi i f(\lambda, \alpha, \sigma+it)} \psi(g(\lambda, \alpha, t)) \sum_{0 \leq n \leq m(t)} \frac{e(\lambda n)}{(n+\alpha)^{\sigma+it}} dt \right| \\ &\ll \sum_{0 \leq n \leq m(T)} (n+\alpha)^{-\sigma} \left| \int_{T_6}^T H(t) e^{ih(t)} dt \right| \end{aligned}$$

with $T_6 := \max \left\{ \frac{T}{2}, 2\pi(n+\alpha)^2 \right\}$ and

$$\begin{aligned} H(t) &:= t^{-\frac{\sigma}{2}} \psi(g(\lambda, \alpha, t)), \\ h(t) &:= -t \log(n+\alpha) - \pi f(\lambda, \alpha, \sigma+it). \end{aligned}$$

Define $l := q(t) - m(t)$, then l is constant 0 or 1. We divide $[T_6, T]$ into subintervals on which l and $q := q(t)$ are constant, namely

$$l = \begin{cases} 1 & , \quad t \in J_1(q) := [2\pi q^2, 2\pi(q + \alpha)^2) \cap [T_6, T] \\ 0 & , \quad t \in J_0(q) := [2\pi(q + \alpha)^2, 2\pi(q + 1)^2) \cap [T_6, T] \end{cases}.$$

Therefore

$$\mathcal{I}_6 \ll \sum_{0 \leq n \leq m(T)} (n + \alpha)^{-\sigma} \sum_{0 \leq l \leq 1} \sum_{n+1-l \leq q \leq q(T)} \left| \int_{J_l(q)} H(t) e^{ih(t)} dt \right|.$$

If $q \leq n + 2$, then

$$\int_{J_l(q)} H(t) e^{ih(t)} dt \ll \int_{2\pi q^2}^{2\pi(q+1)^2} t^{-\frac{\sigma}{2}} dt \ll (q + 1)^{1-\sigma}.$$

Hence

$$\begin{aligned} & \sum_{0 \leq n \leq m(T)} (n + \alpha)^{-\sigma} \sum_{0 \leq l \leq 1} \sum_{n+1-l \leq q \leq n+2} \left| \int_{J_l(q)} H(t) e^{ih(t)} dt \right| \\ & \ll \sum_{0 \leq n \leq m(T)} (n + \alpha)^{1-2\sigma} \ll T^{1-\sigma}. \end{aligned}$$

For $q \geq n + 3$ we will use (4). Therefore, we calculate

$$\begin{aligned} h'(t) &= \log \frac{\left(\frac{t}{2\pi}\right)^{\frac{1}{2}}}{n + \alpha} - \frac{l + \{\lambda\} - \alpha}{2} \left(\frac{t}{2\pi}\right)^{-\frac{1}{2}} \asymp \log \frac{q}{n + \alpha}, \\ h''(t) &= \frac{1}{2t} + \frac{l + \{\lambda\} - \alpha}{8\pi} \left(\frac{t}{2\pi}\right)^{-\frac{3}{2}} \ll q^{-2} \end{aligned}$$

and

$$\begin{aligned} H(t) &\ll q^{-\sigma}, \\ H'(t) &= -\frac{\sigma}{2} t^{-\frac{\sigma}{2}-1} \psi(g(\lambda, \alpha, t)) + t^{-\frac{\sigma}{2}} \psi'(g(\lambda, \alpha, t)) \cdot \frac{1}{2\pi} \left(\frac{t}{2\pi}\right)^{-\frac{1}{2}} \ll q^{-\sigma-1}. \end{aligned}$$

Now (4) leads to

$$\int_{J_l(q)} H(t) e^{ih(t)} dt \ll q^{-\sigma} \left(\log \frac{q}{n + \alpha} \right)^{-1}.$$

Thus,

$$\begin{aligned} & \sum_{0 \leq n \leq m(T)} (n + \alpha)^{-\sigma} \sum_{0 \leq l \leq 1} \sum_{n+3 \leq q \leq q(T)} \left| \int_{J_l(q)} H(t) e^{ih(t)} dt \right| \\ & \ll \sum_{0 \leq n < q-2 \leq q(T)} (n + \alpha)^{-\sigma} q^{-\sigma} \left(\log \frac{q}{n + \alpha} \right)^{-1} \ll T^{1-\sigma} \log T \end{aligned}$$

with the help of Lemma 2. Summing up, we have $\mathcal{I}_6 \ll T^{1-\sigma} \log T$. The integral $\mathcal{I}_7$ can be treated similar. We have

$$\mathcal{I}_7 \ll \sum_{0 \leq n \leq q(T)} (n + \lambda)^{\sigma-1} \left| \int_{T_7}^T K(t) e^{ik(t)} dt \right|$$

with $T_7 := \max \left\{ \frac{T}{2}, 2\pi(n + \lambda)^2 \right\}$ and

$$\begin{aligned} K(t) &:= t^{\frac{1}{2} - \frac{3\sigma}{2}} \psi(g(\lambda, \alpha, t), \\ k(t) &:= -t \log \frac{t}{2\pi e(n + \lambda)} - \pi f(\lambda, \alpha, \sigma + it). \end{aligned}$$

With the same arguments as above one gets after some calculations

$$\begin{aligned} \mathcal{I}_7 &\ll \sum_{0 \leq n \leq q(T)} (n + \lambda)^{\sigma-1} \sum_{0 \leq l \leq 1} \sum_{n+1-l \leq q \leq n+2} q^{2-3\sigma} \\ &\quad + \sum_{0 \leq n \leq q(T)} (n + \lambda)^{\sigma-1} \sum_{0 \leq l \leq 1} \sum_{n+3 \leq q \leq q(T)} q^{1-3\sigma} \log \left(\frac{q}{n + \lambda} \right)^{-1} \\ &\ll \sum_{0 \leq n \leq q(T)} (n + \lambda)^{1-2\sigma} + \sum_{0 \leq n < q-2 \leq q(T)} (n + \lambda)^{\sigma-1} q^{1-3\sigma} \left(\log \frac{q}{n + \lambda} \right)^{-1} \\ &\ll T^{1-\sigma} \log T. \end{aligned}$$

Using the Cauchy-Schwarz inequality and (6) we have

$$\mathcal{I}_8 \ll \left(\int_{\frac{T}{2}}^T t^{\sigma-2} dt \right)^{\frac{1}{2}} \left(\int_{\frac{T}{2}}^T \left| \sum_{0 \leq n \leq m(t)} \frac{e(\lambda n)}{(n + \alpha)^{\sigma+it}} \right|^2 dt \right)^{\frac{1}{2}} \ll \begin{cases} T^{\frac{1}{4}} \log^{\frac{1}{2}} T, & \text{for } \sigma = \frac{1}{2} \\ T^{\frac{\sigma}{2}}, & \text{for } \frac{1}{2} < \sigma < 1. \end{cases}$$

Further,

$$\mathcal{I}_9 \ll \left(\int_{\frac{T}{2}}^T t^{\sigma-2} dt \right)^{\frac{1}{2}} \left(\int_{\frac{T}{2}}^T t^{1-2\sigma} \left| \sum_{0 \leq n \leq q(t)} \frac{e(-\alpha n)}{(n + \lambda)^{1-\sigma-it}} \right|^2 dt \right)^{\frac{1}{2}} \ll T^{\frac{1}{4}} \log^{\frac{1}{2}} T.$$

Finally, one easily gets $\mathcal{I}_{10} \ll 1$. Summing up, we obtain

$$\int_{\frac{T}{2}}^T |L(\lambda, \alpha, 1/2 + it)|^2 dt = \frac{T}{2} \log \frac{T}{\pi} + \frac{T}{2} (c(\alpha) + c(\lambda) - 1) + O(T^{1/2} \log T)$$

and for $1/2 < \sigma < 1$,

$$\begin{aligned} \int_{\frac{T}{2}}^T |L(\lambda, \alpha, \sigma + it)|^2 dt &= \frac{1}{2} \zeta(2\sigma, \alpha) T + \frac{(2\pi)^{2\sigma-1} (1 - 2^{2\sigma-2})}{2 - 2\sigma} \zeta(2 - 2\sigma, \lambda) T^{2-2\sigma} \\ &\quad + O(T^{1-\sigma} \log T + T^{\frac{\sigma}{2}}). \end{aligned}$$

Using the mean square formulas above with $2^{-k}T$ instead of T and adding up over all non-negative integers k , the assertion of the theorem follows. •

4 Applications

In [5]¹ we defined for $0 < \alpha \leq 1, \lambda \in \mathbb{R}, Q \in \mathbb{Z}$ the twist of the Lerch zeta-function $L(\lambda, \alpha, s)$ with a Dirichlet character $\chi \bmod q$ by

$$L(\lambda, \alpha, s, \chi, Q) = \sum_{n=0}^{\infty} \frac{\chi(n+Q)e(\lambda n)}{(n+\alpha)^s} \quad (\sigma > 1).$$

This generalizes Dirichlet L -functions $L(s, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$. Those twists have an analytical continuation throughout the whole complex plane except for at most one simple pole at $s = 1$, which follows immediately from the representation

$$L(\lambda, \alpha, s, \chi, Q) = \frac{1}{q^s} \sum_{0 \leq k < q} \chi(k+Q)e(\lambda k) L\left(\lambda q, \frac{k+\alpha}{q}, s\right).$$

Summing up over all characters $\chi \bmod q$, we get

$$\frac{1}{\varphi(q)} \sum_{\chi \bmod q} |L(\lambda, \alpha, s, \chi, Q)|^2 = \frac{1}{q^{2\sigma}} \sum_{\substack{0 \leq k < q \\ (k+Q, q)=1}} \left| L\left(\lambda q, \frac{k+\alpha}{q}, s\right) \right|^2.$$

Thus, applying the mean square formula, we obtain

Corollary 5 *For $0 < \lambda, \alpha \leq 1$ we have, as T turns to infinity*

$$\begin{aligned} & \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \int_1^T |L(\lambda, \alpha, 1/2 + it, \chi, Q)|^2 dt \\ &= \frac{\varphi(q)}{q} T \log \frac{T}{2\pi} + \frac{T}{q} \left(\varphi(q)c(\lambda q) + \sum_{\substack{0 \leq k < q \\ (k+Q, q)=1}} c\left(\frac{k+\alpha}{q}\right) - \varphi(q) \right) + O\left(T^{\frac{1}{2}} \log T\right) \end{aligned}$$

and for $\frac{1}{2} < \sigma < 1$,

$$\begin{aligned} & \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \int_1^T |L(\lambda, \alpha, \sigma + it, \chi, Q)|^2 dt \\ &= \frac{1}{q^{2\sigma}} \sum_{\substack{0 \leq k < q \\ (k+Q, q)=1}} \zeta\left(2\sigma, \frac{k+\alpha}{q}\right) T + \frac{\varphi(q)}{q^{2\sigma}} \frac{(2\pi)^{2\sigma-1}}{2-2\sigma} \zeta(2-2\sigma, \{\lambda q\}) T^{2-2\sigma} \\ &+ O\left(T^{1-\sigma} \log T + T^{\frac{\sigma}{2}}\right). \end{aligned}$$

¹After the preparation of [5] it becomes known to the authors that Berndt [1] obtained partly similar, and in a joint paper with Schoenfeld [2] in some parts even more general results.

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