SUM OF THE PERIODIC ZETA-FUNCTION OVER THE NONTRIVIAL ZEROS OF THE RIEMANN ZETA-FUNCTION

RAMŪNAS GARUNKŠTIS AND JUSTAS KALPOKAS

ABSTRACT. We consider the asymptotic of the sum of values of the periodic zeta-function $L(s, \lambda)$ over nontrivial zeros of the Riemann zeta-function $\zeta(s)$. J. Steuding for rational λ obtained the main term. Assuming the Generalized Riemann Hypothesis we calculate the next term.

1. INTRODUCTION

Let $s = \sigma + it$ denote a complex variable and define $e(z) = e^{2\pi iz}$. In this paper T always tends to plus infinity and p denotes a prime number. Let $0 < \alpha \leq 1$ and $\lambda \in \mathbb{R}$. The Hurwitz zeta-function is given by

$$\zeta(s, \alpha) = \sum_{n=1}^{\infty} \frac{1}{(n + \alpha)^s} \quad (\sigma > 1)$$

and periodic zeta-function by

$$L(s, \lambda) = \sum_{n=1}^{\infty} \frac{e(\lambda n)}{n^s} \quad (\sigma > 1).$$

For $\alpha = \lambda = 1$ we get the Riemann zeta-function $\zeta(s) = \zeta(s, 1) = L(s, 1)$. The Riemann hypothesis (RH) asserts that all nontrivial (non-real) zeros of $\zeta(s)$ lie on the critical line $\sigma = 1/2$. We will need the generalized Riemann hypothesis (GRH). Consider the Dirichlet L -function

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \quad (\sigma > 1),$$

where $\chi(n)$ is a Dirichlet character. GRH states that in the critical strip $0 < \sigma \leq 1$ the function $L(\chi, s)$ has zeros only on the critical line.

Let $\rho = \beta + i\gamma$ denote the nontrivial zero of the Riemann zeta-function. Assuming RH, Fujii [2] showed

$$(1) \quad \sum_{0 < \gamma \leq T} \zeta(\rho, \alpha) = -\frac{T}{2\pi} \left(\Lambda\left(\frac{1}{\alpha}\right) + L(1, 1 - \alpha) \right) + O(T^{\frac{9}{10}} \log T).$$

Steuding [6] proved the above formula unconditionally with the error term $O(T^{1-c(\log T)^{-2/3}})$, where c is an absolute positive constant. Assuming RH, Garunkštis and Steuding [5] improved Fujii's error term to $O(T^{\frac{1}{2} + \frac{16}{237} + \epsilon})$ and, assuming GRH, this error term can be replaced by $O(T^{\frac{1}{2} + \epsilon})$ if α is rational. For $\alpha = 1/2$ we have

$$\zeta(s, 1/2) = (2^s - 1)\zeta(s).$$

Thus, on RH, nonreal zeros of $\zeta(s, 1/2)$ lie on lines $\sigma = 0$ and $\sigma = 1/2$. In [5] the behavior of sum (1) was investigated when $\alpha = \alpha(T)$ tends to $1/2$ and 1 . This inspired a

Date: November 11, 2007.

AMS 1991 subject classification: 11M35, 11M26

Key words and phrases: periodic zeta-function, Riemann zeta-function, sum over zeros

classification of nontrivial zeros of the Riemann zeta-function dependently on the behavior of zeros of the Hurwitz zeta-function. For more details see [5].

In formula (1) we see the periodic zeta-function. This function is related to the Hurwitz zeta-function by the “functional” equation (Apostol [1], Theorem 12.6)

$$\zeta(1-s, \alpha) = \frac{\Gamma(s)}{(2\pi)^s} \left(e\left(-\frac{s}{4}\right) L(s, \alpha) + e\left(\frac{s}{4}\right) L(s, -\alpha) \right).$$

Let k and l be fixed natural numbers. From Steuding [7] it follows that, for $0 < \frac{k}{l} < 1$, $(k, l) = 1$,

$$\sum_{0 < \gamma \leq T} L\left(\rho, \frac{k}{l}\right) = \left(e\left(\frac{k}{l}\right) - \frac{\mu(l)}{\varphi(l)} \right) \frac{T}{2\pi} \log \frac{T}{2\pi} + O(T).$$

The aim of this paper is to calculate the T term in the above formula.

For $0 < \alpha \leq 1$ we define the generalized Euler constant $\gamma(\alpha)$ by

$$(2) \quad \sum_{0 \leq n \leq T} \frac{1}{n + \alpha} = \log T + \gamma(\alpha) + O\left(\frac{1}{T}\right).$$

We see that $\gamma(1)$ is the usual Euler constant.

For Dirichlet character $\chi \pmod l$ the Gauss sum is defined by

$$\tau(\chi) = \sum_{a=1}^l \chi(a) e\left(\frac{a}{l}\right).$$

By χ_0 we denote the principal character.

The notation $p^v || l$ means that $p^v | l$ but $p^{v+1} \nmid l$.

Theorem 1. *Assume GRH. Let k and l be fixed natural numbers with $0 < \frac{k}{l} < 1$ and $(l, k) = 1$. Then*

$$\sum_{0 < \gamma \leq T} L\left(s, \frac{k}{l}\right) = \left(e\left(\frac{k}{l}\right) - \frac{\mu(l)}{\varphi(l)} \right) \frac{T}{2\pi} \log \frac{T}{2\pi} + C\left(\frac{k}{l}\right) \frac{T}{2\pi} + O(T^{9/10} \log T),$$

where

$$\begin{aligned} C\left(\frac{k}{l}\right) = & -e\left(\frac{k}{l}\right) - \frac{1}{\varphi(l)} \sum_{\substack{\chi \pmod l \\ \chi \neq \chi_0}} \chi(k) \tau(\bar{\chi}) \frac{L'}{L}(1, \chi) \\ & + \frac{\mu(l)}{\varphi(l)} \left(1 + \gamma(1) - \gamma\left(\frac{k}{l}\right) + \sum_{p|l} \frac{\log p}{p-1} \right) - \sum_{p^v || l} \frac{\log p}{1 - p^{-\varphi(\frac{l}{p^v})}} \sum_{j=1}^{\varphi(\frac{l}{p^v})} \frac{e\left(p^j \frac{k}{l}\right)}{p^j}. \end{aligned}$$

In view of

$$L(s, 1/2) = e(-1/2)(1 - 2^{1-s})\zeta(s)$$

the constant $C(\frac{1}{2})$ should be equal to zero. This is indeed, because the sum over characters is empty and $\gamma(\frac{1}{2}) = 2 \log 2 + \gamma(1)$.

To prove the theorem we use Fujii's [2] method. We also make use of the sum

$$(3) \quad \sum_{n=1}^{\infty} \frac{\Lambda(n) - 1}{n} = -2\gamma(1).$$

To calculate this sum, let us consider the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{\Lambda(n) - 1}{n^s}.$$

By the prime number theorem this series converges in the half-plane $\sigma \geq 1$. For $\sigma > 1$,

$$\sum_{n=1}^{\infty} \frac{\Lambda(n) - 1}{n^s} = -\frac{\zeta'}{\zeta}(s) - \zeta(s).$$

Now we take a limit $s \rightarrow 1$.

2. LEMMAS

We will use the approximate functional equation for the periodic zeta-function.

Lemma 1. *If $0 < \lambda \leq 1$, when*

$$L\left(\frac{1}{2} + iT, \lambda\right) = \sum_{1 \leq n \leq \sqrt{\frac{T}{2\pi}}} \frac{e(n\lambda)}{n^{\frac{1}{2} + iT}} + \left(\frac{T}{2\pi}\right)^{-Ti} e^{Ti + \frac{\pi i}{4}} \sum_{0 \leq n \leq \sqrt{\frac{T}{2\pi}}} \frac{1}{(n + \lambda)^{\frac{1}{2} - iT}} + O(T^{-\frac{1}{4}}).$$

Proof. This lemma is the particular case of the more general approximate functional equation for the Lerch zeta-function $L(\lambda, \alpha, s)$, which can be found in Garunkštis, Laurinčikas, and Steuding [4]. Note that $L(s, \lambda) = e(\lambda)L(\lambda, 1, s)$.

In the proof of Theorem 1 the main tool is the following two lemmas of Fujii [2].

Lemma 2. *For $x > 1$ and $T > T_0$ we have*

$$\begin{aligned} \sum_{0 < \gamma \leq T} x^{\gamma i} &= -\frac{T}{2\pi} \frac{\Lambda(x)}{\sqrt{x}} + M(x, T) + O(\sqrt{x} \log(2x)) \\ &+ O\left(\frac{1}{\sqrt{x}} \sum_{\substack{\frac{x}{2} < n < 2x \\ n \neq x}} \Lambda(n) \cdot \min\left(T, \frac{1}{|\log \frac{x}{n}|}\right)\right) \\ &+ O\left(\sqrt{x} \sqrt{\frac{\log T}{\log \log T}}\right) + O\left(x^{\frac{1}{\log \log T}} \log(2x) \frac{\log T}{\log \log T}\right), \end{aligned}$$

where

$$M(x, T) = \begin{cases} \frac{x^{iT} \log(T/2\pi)}{2\pi i \log x} + O\left(\frac{1}{\log x} + \frac{1}{\log^2 x}\right), & \text{if } \frac{1}{\log T} \ll \log x, \\ O\left(\log T \cdot \min\left(\frac{\log T}{\log x}, T \log T\right)\right), & \text{if } \log x \ll \frac{1}{\log T}. \end{cases}$$

Lemma 3. *Suppose that $0 < \frac{2\pi\alpha}{b} \leq Y < T$, $\frac{T}{2\pi\alpha} \gg 1$ and $T > T_0$. Then we have for any positive $b \leq 2$,*

$$\begin{aligned} \sum_{Y < \gamma \leq T} e\left(\frac{b\gamma}{2\pi} \log \frac{b\gamma}{2\pi e\alpha}\right) &= -e^{\frac{\pi i}{4}} \frac{\sqrt{\alpha}}{b} \sum_{\left(\frac{Yb}{2\pi\alpha}\right)^b \leq n \leq \left(\frac{Tb}{2\pi\alpha}\right)^b} \Lambda(n) n^{\frac{1}{2}(\frac{1}{b}-1)} e(-\alpha n^{\frac{1}{b}}) \\ &+ O\left(T^{\frac{2}{5}} \left(\frac{T}{2\pi\alpha}\right)^{\frac{b}{2}}\right) \\ &+ O\left(\left(T^{\frac{1}{2}} \left(\frac{T}{2\pi\alpha}\right)^{-\frac{b}{2}} + Y^{\frac{1}{2}} \left(\frac{Y}{2\pi\alpha}\right)^{-\frac{b}{2}}\right) \log \frac{T}{2\pi\alpha}\right) \\ &+ O\left(\log T \cdot \min\left(\frac{1}{\log \frac{Y}{2\pi\alpha}}, \sqrt{\alpha} + 1\right)\right). \end{aligned}$$

We often will use

Lemma 4. *Assume GRH. If $0 < \frac{k}{l} < 1$ is rational fixed number with $(k, l) = 1$, when*

$$\sum_{n \leq T} \Lambda(n) e\left(n \frac{k}{l}\right) = \frac{\mu(l)}{\varphi(l)} T + O(\sqrt{T} \log^2 T).$$

Proof. We have

$$\sum_{n \leq T} \Lambda(n) e\left(n \frac{k}{l}\right) = \sum_{a=0}^{l-1} e\left(a \frac{k}{l}\right) \sum_{\substack{n \equiv a \pmod{l} \\ n \leq T}} \Lambda(n).$$

Then the lemma follows by (Davenport [3], Chapter 20)

$$\sum_{\substack{n \equiv a \pmod{l} \\ n \leq T}} \Lambda(n) = \frac{1}{\varphi(l)} T + O(\sqrt{T} \log^2 T),$$

and (Apostol [1], Exercise 2.14)

$$\sum_{\substack{a=0 \\ (a,l)=1}}^{l-1} e\left(a \frac{k}{l}\right) = \sum_{\substack{a=1 \\ (a,l)=1}}^l e\left(a \frac{k}{l}\right) = \mu(l).$$

3. PROOF OF THEOREM 1

Using Lemma 1 we have

$$\begin{aligned} (4) \quad \sum_{0 < \gamma \leq T} L\left(\frac{k}{l}, \frac{1}{2} + \gamma i\right) &= \sum_{1 \leq n \leq \sqrt{\frac{T}{2\pi}}} \sum_{2\pi n^2 \leq \gamma \leq T} \frac{e(n \frac{k}{l})}{n^{\frac{1}{2} + \gamma i}} \\ &\quad + e^{\frac{\pi i}{4}} \sum_{0 < \gamma \leq T} e\left(-\frac{\gamma}{2\pi} \log \frac{\gamma}{2\pi e}\right) \left(\frac{k}{l}\right)^{-\frac{1}{2} + \gamma i} \\ &\quad + e^{\frac{\pi i}{4}} \sum_{1 \leq n \leq \sqrt{\frac{T}{2\pi}}} \sum_{2\pi n^2 \leq \gamma \leq T} e\left(-\frac{\gamma}{2\pi} \log \frac{\gamma}{2\pi e}\right) \left(n + \frac{k}{l}\right)^{-\frac{1}{2} + \gamma i} \\ &\quad + O(T^{\frac{3}{4}} \log T) \\ &= S_1 + S_2 + S_3 + O(T^{\frac{3}{4}} \log T). \end{aligned}$$

Term S_1 . By formula (Davenport [3], Chapter 15)

$$\sum_{0 < \gamma \leq T} 1 = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$$

and Lemma 2 we get

$$\begin{aligned}
S_1 &= e\left(\frac{k}{l}\right) \sum_{2\pi \leq \gamma \leq T} 1 + \sum_{2 \leq n \leq \sqrt{\frac{T}{2\pi}}} \frac{e(n\frac{k}{l})}{n^{\frac{1}{2}}} \sum_{2\pi n^2 \leq \gamma \leq T} n^{-i\gamma} \\
&= e\left(\frac{k}{l}\right) \left(\frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} \right) + \sum_{2 \leq n \leq \sqrt{\frac{T}{2\pi}}} \frac{e(n\frac{k}{l})}{n^{\frac{1}{2}}} \left\{ -\frac{T}{2\pi} \frac{\Lambda(n)}{n^{\frac{1}{2}}} + n^2 \frac{\Lambda(n)}{n^{\frac{1}{2}}} - \frac{n^{-iT} \log \frac{T}{2\pi}}{2\pi i \log n} \right. \\
&\quad + \frac{n^{-2\pi n^2 i} \log \frac{2\pi n^2}{2\pi}}{2\pi i \log n} + O\left(\frac{1}{\log n} + \frac{1}{\log^2 n}\right) + O(\sqrt{n} \log(2n)) \\
&\quad + O\left(\frac{1}{\sqrt{n}} \sum_{\frac{n}{2} < m < 2n, m \neq n} \Lambda(n) \cdot \min\left(T, \frac{1}{|\log \frac{n}{m}|}\right)\right) \\
&\quad + O\left(\sqrt{n} \sqrt{\frac{\log T}{\log \log T}}\right) + O\left(n^{1/\log \log T} \log(2n) \frac{\log T}{\log \log T}\right) \\
&\quad \left. - O\left(n^{1/\log \log(2\pi n^2)} \log(2n) \frac{\log(2\pi n^2)}{\log \log(2\pi n^2)}\right) \right\}.
\end{aligned}$$

Using inequalities $\Lambda(n) \leq \log n$ and $|\log(1+x)| \geq |x/2|$ for $-\frac{1}{2} \leq x \leq 1$, similarly as in Fujii [2], we obtain

$$\begin{aligned}
(5) \quad S_1 &= e\left(\frac{k}{l}\right) \left(\frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} \right) - \frac{T}{2\pi} \sum_{2 \leq n \leq \sqrt{\frac{T}{2\pi}}} \Lambda(n) e\left(n\frac{k}{l}\right) \frac{1}{n} \\
&\quad + \sum_{2 \leq n \leq \sqrt{\frac{T}{2\pi}}} \Lambda(n) e\left(n\frac{k}{l}\right) n + O(\sqrt{T} \log^2 T)
\end{aligned}$$

By Lemma 4 and partial summation we see that

$$\sum_{2 \leq n \leq \sqrt{\frac{T}{2\pi}}} \Lambda(n) e\left(n\frac{k}{l}\right) n = \frac{1}{2} \frac{\mu(l)}{\varphi(l)} \frac{T}{2\pi} + O(T^{\frac{3}{4}} \log^2 T).$$

In (5) another sum should be calculated more precisely. Euler's theorem tells us that, for $p^v || l$,

$$(6) \quad p^{\varphi(\frac{l}{p^v})} \equiv 1 \pmod{\frac{l}{p^v}}.$$

This and the orthogonal relation of Dirichlet characters give that

$$\begin{aligned}
& \sum_{2 \leq n \leq \sqrt{\frac{T}{2\pi}}} \Lambda(n) e\left(n \frac{k}{l}\right) \frac{1}{n} \\
&= \sum_{\substack{2 \leq n \leq \sqrt{\frac{T}{2\pi}} \\ (n, l) = 1}} \Lambda(n) e\left(n \frac{k}{l}\right) \frac{1}{n} + \sum_{p|l} \log p \sum_{j=1}^{\infty} \frac{e(p^j \frac{k}{l})}{p^j} \\
&= \frac{1}{\varphi(l)} \sum_{a=1}^l e\left(a \frac{k}{l}\right) \sum_{\chi \pmod l} \overline{\chi(a)} \sum_{n \leq \sqrt{\frac{T}{2\pi}}} \Lambda(n) \frac{1}{n} \chi(n) \\
&+ \sum_{p^v || l} \frac{\log p}{1 - p^{-\varphi(\frac{l}{p^v})}} \sum_{j=1}^{\varphi(\frac{l}{p^v})} \frac{e(p^j \frac{k}{l})}{p^j}.
\end{aligned}$$

If $\chi \neq \chi_0$ then, using the prime number theorem, we obtain

$$\begin{aligned}
\sum_{n \leq \sqrt{\frac{T}{2\pi}}} \Lambda(n) \frac{1}{n} \chi(n) &= \frac{L'}{L}(1, \chi) - \sum_{n > \sqrt{\frac{T}{2\pi}}} \Lambda(n) \frac{1}{n} \chi(n) \\
&= \frac{L'}{L}(1, \chi) + O(T^{-1/4} \log^2 T).
\end{aligned}$$

If $\chi = \chi_0$, then by formula (3)

$$\begin{aligned}
\sum_{n \leq \sqrt{\frac{T}{2\pi}}} \Lambda(n) \frac{1}{n} \chi_0(n) &= \sum_{n=1}^{\infty} \frac{\Lambda(n) \chi_0(n) - 1}{n} - \sum_{n > \sqrt{\frac{T}{2\pi}}} \frac{\Lambda(n) \chi_0(n) - 1}{n} + \sum_{n \leq \sqrt{\frac{T}{2\pi}}} \frac{1}{n} \\
&= \sum_{n=1}^{\infty} \frac{\Lambda(n) - 1}{n} - \sum_{p|l} \sum_{k=1}^{\infty} \frac{\log p}{p^k} + \frac{1}{2} \log \frac{T}{2\pi} + \gamma(1) + O(T^{-1/4} \log^2 T) \\
&= \frac{1}{2} \log \frac{T}{2\pi} - \sum_{p|l} \frac{\log p}{p-1} - \gamma(1) + O(T^{-1/4} \log^2 T).
\end{aligned}$$

From above, in view of

$$\sum_{a=1}^l e\left(a \frac{k}{l}\right) \overline{\chi(a)} = \chi(k) \tau(\overline{\chi}),$$

it follows

$$\begin{aligned}
S_1 &= \left(e\left(\frac{k}{l}\right) - \frac{1}{2} \frac{\mu(l)}{\varphi(l)} \right) \frac{T}{2\pi} \log \frac{T}{2\pi} \\
(7) \quad &+ \left(-\frac{1}{\varphi(l)} \sum_{\substack{\chi \pmod l \\ \chi \neq \chi_0}} \chi(k) \tau(\overline{\chi}) \frac{L'}{L}(1, \chi) - e\left(\frac{k}{l}\right) + \left(\gamma(1) + \frac{1}{2} + \sum_{p|l} \frac{\log p}{p-1} \right) \frac{\mu(l)}{\varphi(l)} \right. \\
(8) \quad &\left. - \sum_{p^v || l} \frac{\log p}{1 - p^{-\varphi(\frac{l}{p^v})}} \sum_{j=1}^{\varphi(\frac{l}{p^v})} \frac{e(p^j \frac{k}{l})}{p^j} \right) \frac{T}{2\pi} + O(T^{3/4} \log^2 T).
\end{aligned}$$

Term S_2 . Using Lemmas 3 and 4 we get

$$\begin{aligned}
 (9) \quad S_2 &= e^{\frac{\pi i}{4}} \left(\frac{k}{l}\right)^{-\frac{1}{2}} \sum_{6\pi \frac{k}{l} \leq \gamma \leq T} e\left(-\frac{\gamma}{2\pi} \log \frac{\gamma}{2\pi e \frac{k}{l}}\right) + O(1) \\
 &= - \sum_{1 \leq n \leq \frac{T}{2\pi \frac{k}{l}}} \Lambda(n) e\left(n \frac{k}{l}\right) + O(T^{\frac{9}{10}}) \\
 &= - \frac{l}{k} \frac{\mu(l)}{\varphi(l)} \frac{T}{2\pi} + O(T^{\frac{9}{10}}).
 \end{aligned}$$

Term S_3 . Again we use Lemmas 3 and 4.

$$\begin{aligned}
 S_3 &= e^{\frac{\pi i}{4}} \sum_{1 \leq n \leq \sqrt{\frac{T}{2\pi}}} \left(n + \frac{k}{l}\right)^{-\frac{1}{2}} \sum_{2\pi n^2 \leq \gamma \leq T} e\left(-\frac{\gamma}{2\pi} \log \frac{\gamma}{2\pi e(n + \frac{k}{l})}\right) \\
 &= - \sum_{1 \leq n \leq \sqrt{\frac{T}{2\pi}}} \sum_{\frac{n^2}{n + \frac{k}{l}} \leq m \leq \frac{T}{2\pi(n + \frac{k}{l})}} \Lambda(m) e\left(m \frac{k}{l}\right) + O(T^{\frac{9}{10}} \log T) \\
 &= - \frac{\mu(l)}{\varphi(l)} \sum_{1 \leq n \leq \sqrt{\frac{T}{2\pi}}} \left(\frac{T}{2\pi(n + \frac{k}{l})} - \frac{n^2}{n + \frac{k}{l}}\right) + O(T^{\frac{9}{10}} \log T) \\
 &= - \frac{\mu(l)}{\varphi(l)} \sum_{1 \leq n \leq \sqrt{\frac{T}{2\pi}}} \left(\frac{T}{2\pi(n + \frac{k}{l})} - \frac{(\frac{k}{l})^2}{n + \frac{k}{l}} - \left(n - \frac{k}{l}\right)\right) + O(T^{\frac{9}{10}} \log T).
 \end{aligned}$$

Then in view of (2) (note that in the definition of the generalized Euler constant the summation starts from zero) we have

$$(10) \quad S_3 = - \frac{1}{2} \frac{\mu(l)}{\varphi(l)} \frac{T}{2\pi} \log \frac{T}{2\pi} + \left(\frac{l}{k} - \gamma\left(\frac{k}{l}\right) + \frac{1}{2}\right) \frac{\mu(l)}{\varphi(l)} \frac{T}{2\pi} + O(T^{\frac{9}{10}} \log T).$$

Now the theorem follows by (4), (7), (9), and (10).

REFERENCES

- [1] T. M. APOSTOL, *Introduction to Analytic Number Theory*, Springer-Verlag, New York, 1976
- [2] A. FUJII, *Zeta Zeros, Hurwitz Zeta Function and $L(1, \chi)$* , Proc. Japan Acad., Ser. A, **65** (1989), 139–142
- [3] G. DAVENPORT, *Multiplicative Number Theory*. Markham, Chicago, 1967
- [4] R. GARUNKŠTIS, A. LAURINČIKAS, AND J. STEUDING, *An approximate functional equation for the Lerch zeta-function*, Math. Notes, **74** (2003), 469–476
- [5] R. GARUNKŠTIS AND J. STEUDING, *On the Distribution of Zeros of the Hurwitz Zeta-function*, Math. Comput., **76** (2007), 323–337
- [6] J. STEUDING, *On the value-distribution of the Hurwitz zeta-function at the nontrivial zeros of the Riemann zeta-function*, Abhdlg. Math. Sem. Uni. Hamburg, **71** (2001), 113–121
- [7] J. STEUDING, *Dirichlet series associated to periodic Arithmetic Functions and the zeros of Dirichlet L -functions*. Anal. Probab. Methods Number Theory, (2002), 282–296

Ramūnas Garunkštis
Department of Mathematics
and Informatics
Vilnius University
Naugarduko 24
2600 Vilnius
Lithuania
ramunas.garunkstis@maf.vu.lt

Justas Kalpokas
Department of Mathematics
and Informatics
Vilnius University
Naugarduko 24
2600 Vilnius
Lithuania
justas.kalpokus@maf.vu.lt